

GAMMA AUTOREGRESSIVE HARMONIC MODEL (GARHM) FOR NON-GAUSSIAN TIME SERIES DATASETS

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B.Sc., M.Sc., (Ibadan)

**A THESIS SUBMITTED IN THE DEPARTMENT OF STATISTICS TO
THE POSTGRADUATE SCHOOL**

**IN PARTIAL FULFIILMENT OF THE REQUIREMENTS FOR THE
AWARD OF DOCTOR OF PHILOSOPHY IN STATISTICS OF OLABISI
ONABANJO UNIVERSITY, AGO-IWOYE, OGUN STATE, NIGERIA**

AUGUST, 2026

ABSTRACT

Seasonality and non-Gaussian properties have been the major concern when analysing and forecasting time series datasets. This is due to the fact that non-Gaussian nature is associated with the excess skewness and seasonality as the conventional Autoregressive and Harmonic models fail to capture datasets with seasonality and non-Gaussian properties. Based on this complexity, a new robust and efficient model called the Gamma Autoregressive Harmonic Model (GARHM) for handling seasonality and non-Gaussian property of the time series datasets has been developed since the transformation changes the structure of the original series. Hence, this study focused on developing, estimating and testing the performance of the Gamma Autoregressive Harmonic Model (GARHM) on datasets from economics, finance and environmental sciences for higher prediction and forecasting.

Exploratory data analysis was performed on Nigeria's monthly exchange rate spanning from January 1996 to February 2025, sourced from an investing website; Nigeria's annual CO₂ emission data from 1990 to 2020, retrieved from Macrotrends website and Nigeria's annual data obtained from the World Bank website on temperature from 1901 to 2023 and inflation from 1961 to 2023. The Gamma Autoregressive Harmonic Model (GARHM) was fitted to the time series datasets. The fitness performance was examined using Coefficient of Determination (R^2), Akaike Information Criterion (AIC), and Bayesian Information Criterion (BIC), while the forecasting evaluation of the developed model was compared using Root Mean Squared Error (RMSE), Mean Absolute Error (MAE), Mean Absolute Scaled Error (MASE), and Mean Absolute Percentage Error (MAPE).

The result showed that the Gamma Autoregressive Harmonic Model (GARHM) outperformed the Autoregressive and Harmonic models as demonstrated by the lowest value of RMSE, MAE, MASE, and MAPE. Empirical analysis of the efficiency of the developed model compared with the existing models was carried out using two sets of financial and environmental time series datasets. The developed GARHM outperformed Autoregressive and Harmonic models, signifying its supremacy. Therefore, the new GARHM recorded higher values of R^2 (0.9813, 0.7256, 0.7985, 0.8992) compared to the autoregressive model (0.9610, 0.4775, 0.6794, 0.6291) and Harmonic model (0.0008671, 0.005610, 0.08315, 0.09075), indicating better performance. Also, the least value of AIC for GARHM (3585.9030, 604.9300, 21.2908, 630.9437) compared to the Autoregressive model (3586.6410, 606.7600, 27.00422, 650.4725) and Harmonic model (4970.9360, 609.4900, 108.3971, 682.9147) for the four datasets implies superiority. The forecasting performance of GARHM has the least RMSE value (39.7097, 14.5380, 0.2410, 3780.2600) compared to the Autoregressive model (39.9794, 27.6210, 0.2580, 7254.2330) and Harmonic model (290.3159, 28.0260, 0.3522, 11358.8330), signifying better predicting power and flexibility.

The developed model showed better fitness and forecasting accuracy compared to the Autoregressive and Harmonic models. This finding is very crucial and important for individuals, stakeholders, and the government for decision-making and policy implementation. Also, the developed model is suitable for datasets from diverse fields with different sample sizes.

Keywords: Gamma Autoregressive Harmonic Model (GARHM), Non-Gaussian, Harmonic Model, Autoregressive, Environmental Sciences

Word Count: 458

CERTIFICATION

This is to certify that this work was carried out by AKANNI Olukunmi Olatunji with matriculation number: PSC/F/21/22/0013 in the Department of Statistics, Olabisi Onabanjo University, Ago-Iwoye, Ogun State under our supervision.

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DEDICATION

This work is dedicated to the Almighty God who has helped me to start and complete this programme with success.

ACKNOWLEDGEMENTS

All thanks and adorations go to the Almighty God who helped me to this extent and will continue to advance and order my steps in life.

I want to appreciate my supervisor, Prof. Timothy O. Olatayo, for his advice, constructive criticism, and encouragement throughout the duration of this project. I am grateful to him for bringing out the skills in me when I encountered challenges. I would also like to appreciate my co-supervisor and Head of Department, Dr. Abass I. Taiwo, for his constructive feedback, words of encouragement, and attention to detail, which have been so helpful in completing this programme. I appreciate the Dean, Faculty of Science, Prof. Olutunde S. Odetunde for his assistance and constructive feedback on the work. I also appreciate all the lecturers in the Department of Statistics, among many are Dr. Biodun T. Efuwape, Dr. Emmanuel F. Ologunleko, Dr. Adedayo F. Adedotun, Dr. Hammed A. Buari, Mr. Adekanmi A. Oyewole and Mr. Ridwan A. Raheem for their support and knowledge kindly shared with me. I also acknowledge all the lecturers in the Department of Mathematics for their contribution and support. My deep appreciation also goes to all non-academic staff in the Department of Statistics and Department of Mathematics for providing a friendly and conducive environment for study and research.

My special thanks go to my wife, Mrs. Akanni Oluwadamilola, and my child, Akanni Oluwafirewamiri, for their support, care, and motivation during the challenging journey of this programme.

I also appreciate my mother, Mrs. Akanni Susan; sisters, Adewuyi Olumide and Akinboyede Tolulope; and brothers, Akanni Olurotimi and Akanni Olukunle, for their support and words of encouragement.

I cannot forget but also thank my late father, Mr. Akanni Johnson, who brought me into this world.

I say a BIG thank you to all.

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ABBREVIATIONS

ACF-	Autocorrelation Function
ADF-	Augmented Dickey-Fuller
AIC-	Akaike Information Criterion
AR-	Autoregressive model
ARIMA-	Autoregressive Integrated Moving Average model
ARIMAX-	Autoregressive Integrated Moving Average Exogenous Variable
AVHRR-	Advanced Very High-Resolution Radiometer
BIC-	Bayesian Information Criterion
GARHM-	Gamma Autoregressive Harmonic Model
GAR-	Gamma Autoregressive
HANTS-	Harmonic Analysis of Time Series
MAE-	Mean Absolute Error
MASE-	Mean Absolute Scaled Error
MAPE-	Mean Absolute Percentage Error
MLE-	Maximum Likelihood Estimation
NDVI-	Distant Normalized Difference Vegetation Index
MOM-	Method of Moment

OLS-	Ordinary Least Square
PACF-	Partial Autocorrelation Function
RMSE-	Root Mean Squared Error

CHAPTER ONE

INTRODUCTION

1.1 Background to the Study

Recorded data over time are common and can be found in any sector ranging from finance, economics, agriculture, ecology, meteorology, and environmental sciences. Time series analysis helps to model data collected over time. It is a powerful technique for decision making (Xu, 2020; Malik *et al.*, 2023; Kolambe, 2024). The method is involved in many real-world applications (He *et al.*, 2022). High-dimensional data is still a rapidly developing field, particularly in the case of dependent data. Forecasting and exploratory and descriptive analysis are the two main areas of research that make up multivariate time series analysis (Wei, 2019). Modeling and understanding the time series variables to effectively represent its dynamic system are the primary goals of the explorative and descriptive analyses (Malik *et al.*, 2023). Yule (1927) created a rigorous mathematical method for time series analysis as an application of autoregressive (AR) to sunspot data.

Spectral analysis is a technique that was first applied in engineering applications to identify periodicities in observed data. Fourier, a mathematician, claimed in 1807 that every series could be roughly represented as the sum of sine and cosine terms (Makridakis, 1976). Schuster (1906) put this theory into practice when he created the term "periodogram" and estimated the length of periodicities concealed behind irregular fluctuations using Fourier expansions. Using the observed data, one can estimate a random signal's power spectrum using a periodogram. It can also be used

as a graphical tool for spectrum analysis, allowing one to determine which periods in the observed series are dominating.

Adedotun *et al.* (2020) proposed a re-parameterized Non-Gaussian Autoregressive (NGAR) model. The result indicated that GAR(2) yielded a more accurate forecast and offer a better model for analyzing GDP growth rate. In the study on the Nigerian monthly exchange rate, NGAR produced efficient estimates and was good for analyzing non-normal data (Adedotun *et al.*, 2022). In a study on high-frequency data, Olatayo and Ekerikevwe (2022) found that log-ARIMAX is more effective and outperforms the conventional ARIMAX model for observations that have heavy-tailed characteristics. Fernandez and Salas (1990) studied the gamma autoregressive (GAR(1)) model for modeling series with a gamma marginal distribution, which produced efficient estimates.

1.2 Statement of the Problem

Normalization has always been done using various transformation techniques as estimation in non-normal distributions poses different challenges. In the past few years, new models applicable to skewed distributions have been developed since the transformation changes the structure of the original series. Harmonic Time Series offer the best method for observation with different harmonic frequencies, which is useful in capturing seasonality and understanding effects of natural and anthropogenic disturbances.

However, these models fail when applied to datasets that exhibit seasonality and Non-Gaussian property as the accurate modeling of the structure of the data is important for effective forecasting and decision-making. Hence, to capture the Non-Gaussian and Seasonality, this study developed a robust model that can captured the presence of these attributes in the datasets.

1.3 Aim and Objectives of the Study

The research aims to develop a Gamma Autoregressive Harmonic Model (GARHM) for Non-Gaussian time series datasets with the following specific objectives:

- (i) use exploratory data analysis for determining the variational pattern of the time series datasets.
- (ii) identify and estimate the parameters of the developed gamma harmonic autoregressive model using maximum likelihood and ordinary least square estimation methods;
- (iii) diagnose and determine the best order for the developed model;
- (iv) compare the results of the developed model with some existing related models.

1.4 Significance of the Study

The Gamma Autoregressive Harmonic model (GARHM) is important and applicable in real-world time series as data violate gaussian property with those coming from environmental sciences, medicine, finance, science, social sciences, and education and show periodic and seasonal patterns. The Gamma Autoregressive Harmonic model (GARHM) involves modeling non-negative or skewed data such as counts, financial returns, rates, and environmental data. By integrating Non-Gaussian, frequency and seasonality, the model can better capture these attributes and produce efficient forecasts. The model was used to analyse data from financial, economic, environmental and meteorological fields. The model provided a sound methodology for estimation of parameters, identification of the model, model diagnosis, and forecasting, which aids decision-making in government and industry. The study addresses the problem encountered by the existing models in

capturing Non-Gaussian, periodic, and seasonal data, as a robust methodology was developed in this study.

Additionally, environmental science and climate change industries greatly benefit, as the model was used to forecast environmental variables like temperature and CO₂ which often show seasonal patterns. The model will help in risk management and portfolio optimization by modeling asset returns. Government and policymakers will greatly benefit from the model, as this will help in analyzing economic indicators like monthly unemployment rates and Gross Domestic Product (GDP) growth.

1.5 Scope of the Study

The issues of Non-Gaussian in environmental and financial time series datasets, particularly when the data exhibits skewness with periodic or seasonal patterns, require an efficient and appropriate method. This research work developed a model that handled Non-Gaussian datasets with periodic or seasonal patterns. The developed model was compared with other methods to validate its efficiency over other models. The model developed was tested with real-life datasets from various disciplines ranging from environmental science, economics and finance.

1.6 Operational Definition of Terms

(i) Stationarity: The constancy in the basic properties of the series.

(ii) Ergodicity: In a stationary time series model, the time average of any realization tends to the mean.

(iii) Non-Gaussian: A series is said to be Non-Gaussian if its probability distributions are asymmetric.

(iv) Harmonic Time series: This is a data set that exhibits periodic or seasonal patterns in its movement.

(v) GARHM: It is a Gamma Autoregressive Harmonic Model and it is a model use to capture datasets that exhibit positive skewness, seasonality, and periodic properties

CHAPTER TWO

LITERATURE REVIEW

2.1 Autoregressive Model

Generally speaking, there are two different kinds of models: general-purpose models and domain-specific models. For comparable (domain) problems, domain-specific models are generally preferred. This regulation is in line with the idea of using what is already known, particularly if the mathematical model already takes this information into account. For instance, whether simulating the endemic spread of a pandemic (Bichara *et al.*, 2014) or radioactive decay (Panchely *et al.*, 2020), for data analysis, a domain-specific model is a useful place to start. General-purpose models can be employed if this domain and use-case background knowledge is unavailable (Ott and Longnecker, 2015).

A thorough theory of the generic linear functional time series was presented by Bosq (2000). The theoretical investigation of weak convergence in nonparametric models was finished by Mas (2007). In nonparametric models, weak convergence is observed. The Fractional Autoregressive (FAR) model tackled the problem of poor convergence for the model's estimates. In order to account for the impact of exogenous functional covariates on the serially dependent curves, Damon and Guillas (2002) suggested the Fractional Autoregressive model with exogenous variables (FARX), and Chen *et al.* (2019b) suggested a partial Fractional Autoregressive (pFAR) model to examine the impact of large-dimensional functional and scalar covariates on the serially dependent functional time series.

Jonas *et al.* (2023) offered an autoregressive modeling method that uses an overparameterized loss function to explicitly account for these uncertainties. The study designs an approach that alternates between state and parameter estimates in order to optimize this loss. Our research demonstrates that the process may effectively rebuild system characteristics and denoise time series. Liu *et al.* (2023) suggested a Skew-T autoregressive model. The paper designed the influence methodology and its validation and used the expectation-maximization (EM) method to estimate its parameters.

A multivariate Beta variate drives the multiplicative evolution of the precision matrix in Uhlig (1997). He indicated that a numerical method based on importance sampling was needed to estimate the autoregressive parameters. In their study of inventory management, Shoesmith and Pinder (2001) contrasted demand projections made with the time series forecasting methods of Vector Autoregressive (VAR) and Bayesian Vector Autoregressive (BVAR) with projections made with seasonal decomposition and exponential smoothing. The research demonstrated that using VAR and Bayesian Vector Autoregressive (BVAR) models can increase forecast accuracy. Using the VAR and the BVAR to characterise the accuracy of inflation and GDP forecasts, Sim (2002) examined the relationship between data and central bank decision-making before comparing restricted and unrestricted VAR models and discussing the role of models and probabilities in the monetary policy process. Sims-Zha Bayesian VAR and unconstrained VAR were utilized by Njenga and Sherris (2011) to model mortality among males and females in a particular community. As their findings showed, Sims-Zha Bayesian VAR was superior.

Adenomon and Oyejola (2014) examined how well the Sims-Zha Bayesian VAR (BVAR) and Reduced Form Vector Autoregression (VAR) forecasted when the independent variables were

collinear, with the BVAR forecast appearing to be better. Dahem (2016) evaluated three models for forecasting inflation by contrasting the standard VAR with the Bayesian VAR. According to the outcome, the Bayesian Vector Error Correction Model (BVECM) was the best model to predict Tunisia's inflation. According to Jadevicius and Huston (2015), ARIMA is a helpful method for evaluating price fluctuations in the broad market. The government and the central bank can forecast the inflation of national home prices using the ARIMA modeling approach. Stevenson (2007) presented the importance of forecasting methods in the stock market, as the ARIMA produced efficient forecasts and helped in predicting stock markets. The lack of reliable statistical techniques makes it difficult to analyze irregular time series. One method is to use interpolation to convert the irregular times into regular ones (Rehfeld *et al.*, 2011). In certain disciplines, negative ACF values are frequently observed (Williams *et al.*, 2006). Significant negative autocorrelation for weekly and monthly stock returns is frequently observed in financial time series (Sewell, 2011). Additionally, stocks with a high trading frequency typically have negative ACF values (Campbell *et al.*, 1997). There hasn't been much discussion of the issue of finding negative ACF values in time series with irregular sampling in the literature. Although it is not always a parsimonious solution.

As an alternative, the continuous-time ARFIMA (CARFIMA) process can fit irregular time series with anti-persistent behavior if it contains an intermediate memory (Tsai, 2009). The work suggested an additional option that relates to an expansion of the IAR model. Sun *et al.* (2007) used the ARFIMA model to examine the data and forecast Great Salt Lake's (GSL) elevation levels in the future. The outcomes demonstrated that the prediction results outperformed the traditional ARMA models. Four models were investigated by Li *et al.* (2007) for the GSL water level. They

discovered that fractional integral autoregressive conditional heteroscedasticity (FIGARCH) performs the best, suggesting that time series with significant volatility should incorporate conditional heteroscedasticity.

Hamjah (2014) forecasted Bangladesh's several seasonal rice productions using an ARIMA model. He came to the conclusion from this investigation that ARIMA models provide accurate forecasts for short-term analysis. Manoj and Madhu (2014) used ARIMA for forecasting. They said that the model was chosen because it takes into account and assumes non-zero autocorrelation; ARIMA (1, 1, 2) was determined to be the optimal model. Additionally, the study statistically examined and confirmed the prediction errors, or consecutive residuals, in the fitted ARIMA model. They came to the conclusion that, according to the ARIMA (2, 1, 0) model, sugarcane production will rise in 2013, then decline in 2014, and then rise overall in the years that follow, up to 2017.

Choudhury and Jones (2014) examined a number of forecasting methods for assessing agricultural output estimates in Ghana in order to offer practical data for economic decision-making. Under each district's data, they examined several models, including the ARMA model, and discovered that it was superior to the other models and more robust (unaffected by cycle duration). Dasyam *et al.* (2015) worked on predicting and modeling India's wheat production between 1961 and 2013. Compared the observed trend in wheat production using three different models: ARIMA, exponential smoothing, and parametric regression. On the basis of multiple goodness of fit criteria, the best model ARIMA (1,1,0) was found, and the predicted value for the 2017–2018 season was

100.271 million tons. Additionally, they looked at the independence and normalcy assumptions using the Shapiro-Wilk and Run tests, respectively.

The random coefficient autoregressive (RCA) and GARCH models were employed by AL-Hakeem and Al-Musawi (2022) and offered a suitable framework for the investigation and evaluation of time-varying volatility and the existence of seasonal effects in financial series. The RCA model and the GARCH model were the best based on the significance of their parameters and AIC value. In order to handle seasonal behaviour in functional time series data, Zamani *et al.*, (2021) developed seasonal functional autoregressive (FSAR) time series models. They presented estimating and prediction techniques, as well as necessary stationarity conditions and limiting behaviour for the model of order one.

Aknouche *et al.* (2017) noted that periodic vector autoregressive (PVAR) models have a lot of parameters, which makes them quite adaptable. However, because of severe overfitting problems, unconstrained PVAR estimation frequently performs poorly in applications with moderate sample sizes, as is commonly seen in macro econometrics. Ursu and Duchesne (2008); and Boubacar and Ursu (2023) developed least squares estimators with linear parameter constraints to address these issues. But they only permit the imposition of linear constraints on parameters within a specified season to control the number of parameters to be estimated. Tsiakas (2006) introduced a number of intriguing and economical time-varying stochastic volatility models to explain periodicity in the volatility. These models use suitable exogenous variables and deterministic periodic functions of time to represent the volatility parameters. The suggested periodic stochastic volatility (PSV) models have been effectively used to simulate the daily S&P 500 return development. This

suggests that time series volatility may be characterized by a periodically shifting structure. However, the PSV formulations may overlook a potential stochastic periodicity in these moments because they are particularly well suited to a type of deterministic periodicity in the second moment.

2.2 Gamma Autoregressive Model

The most popular mathematical models in hydrology for this purpose are the autoregressive time series (AR) models. A number of presumptions form the basis of these models. The series of error terms' normal distribution is the most crucial presumption, as this assumption is mostly based on the challenges of estimating model parameters. AR models that work with non-normal distributions have also been created recently.

Sim (1987) thought of a time series model that might be used to simulate the long-term correlation structure of an ARMA and stationary river flow sequences with strong skewness (1,1). The method of moments (MOM) was used under gamma distribution. Using this model, 100 monthly streamflow sequences were simulated. The gamma autoregressive (GAR(1)) model was examined by Fernandez and Salas (1990). Prior to utilizing MOM to estimate the parameters, some sort of correction was suggested because the computed parameters are skewed for dependent and non-normal variables. When the suggested approach was used to create synthetic streamflow data from certain annual streamflow data, it was asserted that the GAR(1) model was more realistic than earlier ones.

Cıgızoğlu and Bayazit (1998) applied the GAR(1) model to ten Turkish rivers' annual flow series. They discovered that bias modifications were crucial in determining the run and range parameters.

Gamma innovations in the AR(1) model were examined by Tiku *et al.* (1999). They recommended the modified maximum likelihood (MML) approach. It is possible to approximate annual streamflow series to normal distributions, according to Fernandez and Salas (1986). However, this is no longer the case for brief time periods, especially due to the issue of skewness. A number of approaches have been tried to approximate a normal distribution in order to get around this problem, but none of them have been successful. Many of these models assume a non-normal residuals and feature first-order autoregressive structures that are either constant or periodic for all of the data in the series. By utilizing an autoregressive structure and a gamma distribution, Fernandez and Salas (1986) presented a model that addressed the aforementioned constraints, they developed a hybrid model that combined the features of additive and multiplicative formulations. The study failed not investigate higher orders due to the complexity of the model, which would require the construction of a multivariate gamma distribution.

Fernandez (1984) used the same concepts and mathematical techniques as order model 1 to explain how to simply construct the periodic additive gamma autoregressive (PAGAR) model for order 2. In addition to the mathematical complexity of the model, the author highlights the difficulties in assigning the exponential marginal and the difficulty in reducing PAGAR(2) to the specific scenario when the asymmetry coefficient is constant for all periods. Because of these limitations, the order 2 model is not very useful. Chebaane *et al.* (1995) used the periodic gamma autoregressive (PGAR) model in conjunction with the periodic discrete autoregressive (PDAR) model to model hydrological series in arid regions. Fernandez and Salas (1990) suggested a technique for correcting the bias in estimates and called the model PGAR–PDAR.

In the framework of the Brazilian electrical industry (BES), Ferreira (2013) discussed the hybrid model as a substitute for the periodic autoregressive (PAR(p)) model for creating synthetic series.

Braga and Calmon (2017) contrasted the PGAR results with the lognormal and periodic normal models used in the industry. They discovered that the PGAR model was a good substitute for the lognormal model and produced better results when compared to the periodic normal model. In order to create synthetic energy series for long-term operational planning for thermal and hydroelectric systems, Marcato *et al.* (2012) suggested a methodology based on PAR(p). Furthermore, Bernard and Abdeslem (1996) address the modeling of hydroelectric systems' long-term operations, which is an application of Markov decision processes. Fuzzy logic has already been suggested for long-term studies by Torrini *et al.* (2016) in order to derive rules from the input variables and produce annual energy demand projections. The features of the PAR(p) model, which is utilized to create synthetic NH series that are inputs to computer platforms used to carry out operations planning and extension in the BES, are covered by Ferreira *et al.*, (2015b).

2.3 Harmonic Time Series

Electric power distribution networks' harmonic sources are examined. The study looks at feeders in a particular electrical system using basic linear regression analysis. Using this method, the study evaluates the impact of nonlinear loads in a feeder on a bus within the electrical system. Field measurements were used for the analysis throughout a seven-day period (Almeida *et al.*, 2019). A partial least-squares regression was presented by Yonghai *et al.* (2006) for assessing emission levels and harmonic impedance at the point of common coupling (PCC); the method was used to calculate regression coefficients.

Yin *et al.* (2011) used partial correlation analysis for voltage and current harmonic distortions along with the multiple linear regression method to quantify the harmonic influence on numerous

loads. Using harmonic analysis, Tarawneh and Kadioglu (2002) findings revealed the same outcome with Kadioglu *et al.* (1999) in terms of the first harmonic's percentage of variance, which accounts for over 90% of variation. This is because Jordan and Turkey share a Mediterranean climate and are impacted by comparable weather patterns. Because of anthropogenic forcing and the time obtained from equinoxes, estimates of the yearly seasonal cycle differ between land and ocean (Donohoe *et al.*, 2020; Santer, 2018). There are various kinds of time series recordings of climate variables that are accessible in meteorological stations. Seasons in various time series within an annual cycle, however, must be distinct. Therefore, it seems sensible to anticipate that the seasons in time series of various climate variables would be comparable. As a result, this work used harmonic analysis, which has been tried and tested in a number of geoscientific applications (Padhee and Dutta, 2019; Tarawneh, 2016), to estimate the seasonal limits.

In order to explain seasonal trends in global surface air temperature, Legates and Willmott (1990) used harmonic analysis. They distinguished between regions with a strong biannual cycle in air temperature and regions with a strong seasonal cycle in air temperature. The analysis has been shown in the majority of the contributions to be a useful tool for examining seasonal temperature cycles as well as variations in daily and annual rainfall (Zhang *et al.*, 2017). The missing and distant normalized difference vegetation index (NDVI) time series data are also reconstructed using the HANTS technique. The effectiveness of the harmonic analysis of time series (HANTS) method in reconstructing MODIS NDVI time series was assessed by Zhou *et al.*, (2015). Using the HANTS technique, Jiang *et al.*, (2008) recreated the impact of cloud cover on the Advanced Very High-Resolution Radiometer (AVHRR) NDVI time series from 1981 to 2001. A study investigated the HANTS algorithm's ability to recreate NDVI time series spanning one and five

years in a different study (Hosseini *et al.*, 2017). These studies' findings demonstrated that the HANTS algorithm can be used to successfully address the issue of missing data in NDVI time series analysis.

Wen *et al.* (2004) investigated whether HANTS's cloud removal might be used to rebuild ten days' worth of composited AVHRR data across the Tibetan Plateau's mountainous areas. Wit and Su (2005) looked into HANTS's capacity to interpret NDVI time series from SPOT-VEGETATION and determine when the growth season begins and ends. Furthermore, De Jong *et al.* (2011) assessed the global applicability of HANTS in their analysis of global browning and greening patterns using the GIMMS NDVI datasets. Julien *et al.* (2006) estimated vegetation changes in the European continent by processing land surface temperature (LST) and NDVI time series datasets simultaneously. This study emphasizes the removal of clouds and the extraction of quantitative phenological information by HANTS. Once again, leveraging these two capabilities, Alfieri *et al.* (2013) created a method to map air temperature using daily LST products from MODIS, and Jia *et al.* (2011) used NDVI time series from MODIS to analyze the phenological response of vegetation to upstream river flow in the Heihe River Basin of China. However, for cropland classification and phenology monitoring, respectively, Jakubauskas *et al.*, (2002) and Geerken (2009) used the amplitude and phase of harmonic components provided by harmonic analysis without the cloud detection procedure of HANTS. Zhou *et al.* (2013) used random gap circumstances and a variety of reference NDVI series to simulate HANTS's performance. In order to increase generality, the NDVI time series were built using real NDVI data from various ecosystems and noise patterns. Additionally, the correctness of the reconstruction using HANTS of a worldwide NDVI dataset was evaluated using these quantitative results (Zhou *et al.*, 2013).

Qiang *et al.* (2022) aimed to tackle issues in harmonic time series analysis for seasonal land surface dynamics by utilizing remote sensing data. These challenges encompass the complexity of frequency selection, susceptibility to noise, and missing data due to atmospheric interference. They presented the Harmonic Adaptive Penalty Operator (HAPO), a regression method featuring a novel penalty function designed to reduce unforeseen variations and overfitting in time series models (Friedman *et al.*, 2010). The research demonstrated that the Harmonic Adaptive Penalty Operator (HAPO) consistently surpassed conventional regression techniques in harmonic time series analysis, especially in managing irregularly distributed data and time gaps. The Root Mean Square Deviation (RMSD) values for HAPO generally stayed under 0.04, demonstrating its reliability in reflecting seasonal land surface changes despite variations in data density. Analysis of seasonal variations highlighted the stability of HAPO. Both HAPO and Ridge regression showed stable RMSD values throughout various seasons, even in areas with difficult climate conditions like Alaska and North Dakota. Conversely, OLS and LASSO exhibited higher RMSD values, especially during times with less distinct observations. This consistency across various situations highlights HAPO's dependability and flexibility in harmonic time series analysis.

Mark and David (2000) utilized harmonic analysis on a one-year time series of NOAA-AVHRR NDVI (Normalized Difference Vegetation Index) composite data, which included 26 biweekly intervals. This method breaks down periodic time-varying phenomena into sinusoidal functions (Eidenshink, 1992). Images of amplitude and phase angle obtained from the analysis were subsequently correlated with data on crop type and extent to develop a method for identifying and classifying various crop types (Egbert *et al.*, 1995). The research emphasized its possible

advantages in various domains, such as mapping land use and cover, observing landscape stability and change, determining crop types, and minimizing noise and data volume in time-series images. The findings showed that harmonic analysis could successfully identify and distinguish seasonal variations in NDVI across different land-use and land-cover categories. Images of amplitude and phase angle obtained from the analysis showed a strong correlation with crop type and extent, facilitating precise identification of crop types. This method also minimized noise and data complexity in the time-series images, improving its effectiveness for remote sensing uses. The results indicate that harmonic analysis is especially useful for crops with clear phenological traits, offering important understanding of seasonal patterns and enhancing land-use observation.

Cao *et al.* (2015) sought to identify the ideal length of time series (TS) for harmonic analysis (HA) while simulating various tidal constituents, specifically the M2, S2, K1, and O1 tidal constituents in the Bohai and Yellow Seas. To accomplish this, a two-dimensional tidal model was utilized to simulate the four main tidal constituents. Harmonic analysis was conducted on multiple nonoverlapping time series of different lengths to determine the field-average mean variance deviation (AMVD). The AMVD represents the typical variation of HA results throughout the field, and a sufficiently minor AMVD suggests that the HA results do not depend on the TS length. The associated TS length was subsequently considered as the ideal (Zu *et al.*, 2008). The analysis revealed that the ideal TS length for HA in simulating the four main tidal constituents varied between 180 and 195 days. Within this interval, the AMVD was adequately minimal, indicating that HA outcomes were reliable and not affected by the selection of TS length. The research indicated that the ideal length was not influenced by elements like advection, nodal corrections, and calculated area. Rather, the main factor was bottom friction. When linear bottom friction was

utilized, any TS duration exceeding 15 days was appropriate for HA. Nonetheless, nonlinear bottom friction led to the introduction of higher harmonics, requiring a TS duration of 180–195 days to obtain optimal outcomes. This discovery emphasizes the considerable influence of bottom friction on harmonic analysis and stresses the necessity of taking this aspect into account when establishing the TS length for precise tidal simulations.

Young *et al.* (1999) illustrated the efficacy of the suggested Dynamic Harmonic Regression (DHR) model and its related optimization technique in enhancing the analysis of nonstationary time series. An important result is the improved convergence characteristics and computational efficiency obtained via the frequency-domain cost function. This method surpassed conventional Maximum Likelihood (ML)-based techniques by offering a clearly defined cost function that greatly decreased numerical requirements while preserving precision in component estimation. Additionally, the DHR model proved effective in managing modulated cycles and trigonometric seasonality, highlighting its adaptability in tackling periodic dynamics. This flexibility renders the model especially significant for systems that necessitate a detailed examination of seasonal and cyclical relationships.

Anzhou *et al.* (2015) conducted a study to examine the ideal duration of time series (TS) needed for harmonic analysis (HA) in modeling various tidal constituents in the Bohai and Yellow Seas. They examined multiple distinct time series of different lengths, spanning from 15 to 365 days. For every TS length, the average deviation of the HA results across the field was determined. A sufficiently minor deviation suggested that the HA results were uniform across TS lengths, identifying the corresponding TS length as the best choice. The research determined that a TS

length of 180–195 days is ideal for accurately modeling. Further experiments were carried out employing various model configurations and calculated areas. The results indicated that the ideal TS length did not depend on advection, nodal adjustments, or the calculated area. Rather, it was mainly affected by bottom friction. In particular, nonlinear bottom friction was shown to produce higher harmonics, requiring a TS length of 180–195 days to yield precise HA results. Conversely, with the application of linear bottom friction, a TS length exceeding 15 days was adequate for dependable harmonic analysis.

Angel and Fernando (2022) conducted a study to model time series using a univariate and multivariate harmonic decomposition approach. The approach utilizes the Kalman-Bucy filter as an ideal state observer. Throughout the convergence phase, the observer verifies that the states (harmonics) stabilize, allowing the model to predict the signal accurately (Chen *et al.*, 2005). This enables time series prediction after the observer model attains stability. The method can be modified for use in both univariate and multivariate situations. In a multivariate context, statistical analysis is used to discover significant variables that improve model precision, enabling the inclusion of various influencing elements. The findings of this research show the efficacy of the suggested univariate and multivariate harmonic decomposition approach in modeling and predicting time series data. The approach, based on Fourier series analysis, effectively captured the harmonic components of signals and transformed them into a time-dependent linear state space model. In the univariate scenario, the model was utilized on the time series of electrical power consumption. The findings showed that the harmonic decomposition method effectively recognized the periodic elements of the data, resulting in trustworthy short- and medium-term predictions. The convergence of the state observer underscored the model's ability to adjust to the

varying traits of the power demand signal, emphasizing its value in energy management systems. In the multivariate application, the model was updated to include a wind/solar energy profile. Through a statistical analysis aimed at identifying the most significant variables, the multivariate framework improved the model's predictive precision. The addition of extra variables enabled the methodology to consider interdependencies and fluctuations in the renewable energy data, leading to predictions that closely matched the observed trends.

Farhana *et al.* (2012) examined the effects of harmonics through measured harmonic data, focusing on the increasing worry about harmonic distortion from nonlinear customer loads. The findings from the simulations demonstrate the substantial effect of harmonic loads on the distribution system. In the case study devoid of harmonic loads, the power quality indicators stayed within acceptable ranges, indicating consistent system performance. Nonetheless, the incorporation of harmonic loads in the second case study revealed a significant rise in harmonic distortion levels, especially within the voltage and current waveforms. The time-series simulation revealed insights into the temporal fluctuations of harmonic distortion, indicating that maximum distortion levels aligned with times of increased load activity in commercial properties. This discovery highlights the necessity of ongoing monitoring and time-series analysis to understand the evolving characteristics of harmonic effects in distribution systems.

CHAPTER THREE

MATERIALS AND METHODS

This chapter discusses the developed gamma autoregressive harmonic model (GARHM) for Non-Gaussian time series datasets, which can be applied in different disciplines.

3.1 Autoregressive Model

An autoregressive model is defined as

$$y_t = \varphi_0 + \varphi_1 \sum_{i=1}^p \varphi_i y_{t-i} + \varepsilon_t \quad (3.1)$$

where

y_t represents univariate variable with constant interval of the time observations.

y_{t-1} represents univariate variable with constant interval of the time observations with lag i .

φ_0 represents autoregressive constant

φ_1 represents autoregressive coefficient with different lag

$\varepsilon_t \sim N(0, \sigma^2)$

3.2 Gamma Model

$$y_t = \frac{\phi_2^{\phi_1}}{\Gamma(\phi_2)} y_t^{\phi_1-1} e^{-\phi_2 y_t} \quad (3.2)$$

where

y_t represents univariate variable with constant interval of the time observations.

ϕ_1 represents shape parameter

ϕ_2 represents scale parameter

3.2.1 Parameter Estimation of Gamma Model

Likelihood Function:

$$L_k(\phi_1, \phi_2) = \prod_{t=1}^n P(y_t) = \prod_{t=1}^n f(y_t) = \prod_{t=1}^n \frac{\phi_2^{\phi_1}}{\Gamma(\phi_1)} y_t^{\phi_1-1} e^{-\phi_2 y_t} \quad (3.3)$$

Log-Likelihood Function:

$$L(\phi_1, \phi_2) = \sum_{i=1}^n \log \left[\frac{\phi_2^{\phi_1}}{\Gamma(\phi_1)} y_{ti}^{\phi_1-1} e^{-\phi_2 y_{ti}} \right] \quad (3.4)$$

Expanding and simplifying:

$$L(\phi_1, \phi_2) = n\phi_1 \log(\phi_2) - n \log(\Gamma(\phi_1)) + (\phi_1 - 1) \sum_{i=1}^n \log(y_{ti}) - \phi_1 \sum_{i=1}^n y_{ti} \quad (3.5)$$

$$\frac{\partial L(\phi_1, \phi_2)}{\partial \phi_1} = \frac{\partial}{\partial \phi_1} \left[n\phi_1 \log(\phi_2) - n \log(\Gamma(\phi_1)) + (\phi_1 - 1) \sum_{i=1}^n \log(y_{ti}) - \phi_1 \sum_{i=1}^n y_{ti} \right] \quad (3.6)$$

$$= n \log(\phi_2) - n \frac{\Gamma'(\phi_1)}{\Gamma(\phi_1)} + \sum_{i=1}^n \log(y_{ti}) \quad (3.7)$$

$$\frac{\partial L(\phi_1, \phi_2)}{\partial \phi_2} = \frac{\partial}{\partial \phi_2} \left[n\phi_1 \log(\phi_2) - n \log(\Gamma(\phi_1)) + (\phi_1 - 1) \sum_{i=1}^n \log(y_{ti}) - \phi_2 \sum_{i=1}^n y_{ti} \right] \quad (3.8)$$

$$= n \frac{\phi_1}{\phi_2} - \sum_{i=1}^n y_{ti} \quad (3.9)$$

From Equation (3.9):

$$\frac{\partial L(\phi_1, \phi_2)}{\partial \phi_2} = \frac{n\phi_2}{\phi_1} - \sum_{i=1}^n y_{ti} = 0 \quad (3.10)$$

Solving for ϕ_2 :

$$\hat{\phi}_2 = \frac{n\phi_1}{\sum_{t=1}^n y_t} \quad (3.11)$$

Let $\bar{X} = \frac{1}{n} \sum_{i=1}^n X_{ti}$:

$$\hat{\phi}_2 = \frac{\phi_1}{\bar{y}} \quad (3.12)$$

$$\frac{\partial L(\phi_1, \phi_2)}{\partial \phi_2} = 0$$

$$n \log(\phi_2) - n \frac{\Gamma'(\phi_1)}{\Gamma(\phi_1)} + \sum_{i=1}^n \log(y_{ti}) = 0 \quad (3.13)$$

$$\log(\phi_1) - \log(\bar{y}) - \frac{\Gamma'(\phi_1)}{\Gamma(\phi_1)} + \frac{1}{n} \sum_{i=1}^n \log(y_{ti}) = 0 \quad (3.14)$$

Equation (3.14) is a nonlinear equation that does not exist in closed form. Therefore, the equation can be solved numerically using the Newton-Raphson Method. The score function is denoted by $U(\hat{\phi}_2; y_{t1}, y_{t2}, \dots, y_{tn})$. Then, the MLE $\hat{\phi}_2$ is the solution of:

$$U(\phi_2) = \frac{\partial L(\phi_1, \phi_2)}{\partial \phi_2} = 0 \quad (3.15)$$

$$\Rightarrow U(\phi_2^*) = 0$$

The equation (3.15) can be approximated using the Newton-Raphson Method:

$$U(\phi_2^*) \approx U(\phi_2^{(t)}) + U'(\phi_2^{(t)})(\phi_2^* - \phi_2^{(t)}) \quad (3.16)$$

Using Equation (3.16) into (3.15), then solve for ϕ_2^* :

$$U(\phi_2^{(t)}) + U'(\phi_2^{(t)})(\phi_2^* - \phi_2^{(t)}) = 0 \quad (3.17)$$

Rearranging:

$$\phi_2^* = \phi_2^{(t)} - \frac{U(\phi_2^{(t)})}{U'(\phi_2^{(t)})} \quad (3.18)$$

Given the initial $\phi_2^{(1)}$, the iteration converges when:

$$\phi_2^{(t+1)} = \phi_2^{(t)} - \frac{U(\phi_2^{(t)})}{U'(\phi_2^{(t)})} \quad (3.19)$$

3.3 Fitting Sinusoids and Estimation of Parameters for Harmonic Model

$$y_t = \beta \sin(\omega_t) + \gamma \cos(\omega_t) + \varepsilon_t \quad (3.20)$$

y_t represents univariate variable with constant interval of the time observations.

β and γ represent the coefficients for the sinusoidal components

ω represents different frequencies for each harmonic component

$$\varepsilon_t \sim N(0, \sigma^2)$$

The least squares estimation is:

$$\varepsilon_t = S(\beta, \gamma) = \sum_{i=0}^{n-1} (Y_t - \beta \sin(\omega t) - \gamma \cos(\omega t))^2 \quad (3.21)$$

$$= \sum_{i=0}^{n-1} (y_t - \gamma \cos(\omega t) - \beta \sin(\omega t))^2 \quad (3.22)$$

Differentiating with respect to γ :

$$\frac{\partial \varepsilon_t}{\partial \gamma} = -2 \sum \cos(\omega t) (y_t - \gamma \cos(\omega t) - \beta \sin(\omega t)) = 0 \quad (3.23)$$

Differentiating with respect to β :

$$\frac{\partial \varepsilon_t}{\partial \beta} = -2 \sum \sin(\omega t) (y_t - \gamma \cos(\omega t) - \beta \sin(\omega t)) = 0 \quad (3.24)$$

Solving for γ :

$$\gamma = \hat{\gamma} = \frac{1}{\Delta} \left\{ \sum y_t \cos(\omega t) \sum (\sin(\omega t))^2 - \sum y_t \sin(\omega t) \sum \cos(\omega t) \sin(\omega t) \right\} \quad (3.25)$$

Solving for β :

$$\beta = \hat{\beta} = \frac{1}{\Delta} \left\{ \sum y_t \sin(\omega t) \sum (\cos(\omega t))^2 - \sum Y_t \cos(\omega t) \sum \cos(\omega t) \sin(\omega t) \right\} \quad (3.26)$$

Where:

$$\Delta = \sum (\cos(\omega t))^2 \sum (\sin(\omega t))^2 - \left(\sum \cos(\omega t) \sin(\omega t) \right)^2 \quad (3.27)$$

The sums involving trigonometrical functions can be calculated as:

$$\sum_{t=0}^n (\cos(\omega t))^2 = \frac{n}{2} (1 + \Delta_n(2f) \cos(2\pi(n-1)f)) \quad (3.28)$$

$$\sum_{t=0}^n \cos(2\pi f t) \sin(2\pi f t) = \frac{n}{2} \Delta_n(2f) \sin(2\pi(n-1)f) \quad (3.29)$$

$$\sum_{t=0}^n (\sin(2\pi f t))^2 = \frac{n}{2} (1 - \Delta_n(2f) \cos(2\pi(n-1)f)) \quad (3.30)$$

$$\Delta = \frac{n^2}{4} (1 - \Delta_n(2f)^2) \quad (3.31)$$

where:

$$\Delta_n = \frac{\sin(\pi f n)}{n \sin(\pi f)}$$

The characteristics of the trigonometric functions are given as:

$$\sum_{t=0}^{n-1} \cos(2\pi f_j t) = 0, \text{ if } j \neq 0 \quad (3.32)$$

$$\sum_{t=0}^{n-1} \sin(2\pi f_j t) = 0, \text{ if } j \neq 0 \quad (3.33)$$

$$\sum_{t=0}^{n-1} \cos(2\pi f_j t) \sin(2\pi f_{j'} t) = \begin{cases} \frac{n}{2}, & j = j' \neq 0 \text{ or } \frac{n}{2}, \\ n, & j = j' = 0 \text{ or } \frac{n}{2}, \\ 0, & j \neq j', \end{cases} \quad (3.34)$$

$$\sum_{t=0}^{n-1} \sin(2\pi f_j t) \cos(2\pi f_{j'} t) = \begin{cases} \frac{n}{2}, & j = j' \neq 0 \text{ or } \frac{n}{2}, \\ 0, & \text{otherwise.} \end{cases} \quad (3.35)$$

$$\text{Var}(\hat{\varepsilon}) = \frac{\sigma^2}{n}$$

$$\text{Var}(\hat{\gamma}) = \text{Var}(\hat{\beta}) = \frac{\partial \sigma^2}{\Delta}$$

$$\text{Cov}(\hat{\varepsilon}, \hat{\gamma}) = \text{Cov}(\hat{\varepsilon}, \hat{\beta}) = \text{Cov}(\hat{\gamma}, \hat{\beta}) = 0$$

3.4 Gamma Autoregressive Harmonic Model

The Gamma Autoregressive Harmonic Model of order 1 is given as

$$y_t = \phi y_{t-1} + \beta \sin(\omega t) + \gamma \cos(\omega t) + \varepsilon_t \quad (3.36)$$

y_t represents univariate variable with constant interval of the time observations.

y_{t-1} represents univariate variable with constant interval of the time observations with lag i .

β and γ represent the coefficients for the sinusoidal components

ω represents different frequencies for each harmonic component

ε_t represents error term, typically assumed to follow gamma distribution.

$\sin(\omega t)$ and $\cos(\omega t)$ represent seasonal components.

ϵ_t represents error term, typically assumed to be normally distributed, $\mathcal{N}(0, \sigma^2)$.

3.4.1 Maximum Likelihood Estimation of Gamma Autoregressive Harmonic Model

$$f(Y_t|Y_{t-1}, \sin(\omega t), \cos(\omega t), \phi, \beta, \gamma, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(Y_t - \phi Y_{t-1} - \beta \sin(\omega t) - \gamma \cos(\omega t))^2}{2\sigma^2}\right) \quad (3.37)$$

$$L(\phi, \beta, \gamma,) = \prod_{t=1}^n \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(Y_t - \phi Y_{t-1} - \beta \sin(\omega t) - \gamma \cos(\omega t))^2}{2\sigma^2}\right) \quad (3.38)$$

Taking natural log of (3.38)

$$\ell(\phi, \beta, \gamma,) = -\frac{n}{2} \log(2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum_{t=1}^n (Y_t - \phi Y_{t-1} - \beta \sin(\omega t) - \gamma \cos(\omega t))^2 \quad (3.39)$$

$$\frac{\partial \ell}{\partial \phi} = \frac{1}{\sigma^2} \sum_{t=1}^n (Y_t - \phi Y_{t-1} - \beta \sin(\omega t) - \gamma \cos(\omega t)) Y_{t-1} \quad (3.40)$$

$$\sum_{t=1}^n Y_t Y_{t-1} - \phi \sum_{t=1}^n Y_{t-1}^2 - \beta \sum_{t=1}^n \sin(\omega t) Y_{t-1} - \gamma \sum_{t=1}^n \cos(\omega t) Y_{t-1} = 0 \quad (3.41)$$

$$\phi = \frac{\sum_{t=1}^n Y_t Y_{t-1} - \beta \sum_{t=1}^n Y_{t-1} \sin(\omega t) - \gamma \sum_{t=1}^n Y_{t-1} \cos(\omega t)}{\sum_{t=1}^n Y_{t-1}^2} \quad (3.42)$$

$$\hat{\phi} = \frac{\sum_{t=1}^n (Y_t - \hat{\beta} \sin(\omega t) - \hat{\gamma} \cos(\omega t)) Y_{t-1}}{\sum_{t=1}^n Y_{t-1}^2} \quad (3.43)$$

$$\frac{\partial \ell}{\partial \beta} = \frac{1}{\sigma^2} \sum_{t=1}^n (Y_t - \phi Y_{t-1} - \beta \sin(\omega t) - \gamma \cos(\omega t)) \sin(\omega t) \quad (3.44)$$

$$\sum_{t=1}^n Y_t \sin(\omega t) - \phi \sum_{t=1}^n Y_{t-1} \sin(\omega t) - \beta \sum_{t=1}^n \sin^2(\omega t) - \gamma \sum_{t=1}^n \sin(\omega t) \cos(\omega t) = 0 \quad (3.45)$$

$$\hat{\beta} = \frac{\sum_{t=1}^n (Y_t - \hat{\phi} Y_{t-1} - \hat{\gamma} \cos(\omega t)) \sin(\omega t)}{\sum_{t=1}^n \sin^2(\omega t)} \quad (3.46)$$

$$\frac{\partial \ell}{\partial \gamma} = \frac{1}{\sigma^2} \sum_{t=1}^n (Y_t - \phi Y_{t-1} - \beta \sin(\omega t) - \gamma \cos(\omega t)) \cos(\omega t) \quad (3.47)$$

$$\sum_{t=1}^n Y_t \cos(\omega t) - \phi \sum_{t=1}^n Y_{t-1} \cos(\omega t) - \beta \sum_{t=1}^n \sin(\omega t) \cos(\omega t) - \gamma \sum_{t=1}^n \cos^2(\omega t) = 0 \quad (3.48)$$

$$\hat{\gamma} = \frac{\sum_{t=1}^n (Y_t - \hat{\phi} Y_{t-1} - \hat{\beta} \sin(\omega t)) \cos(\omega t)}{\sum_{t=1}^n \cos^2(\omega t)} \quad (3.49)$$

3.4.2 Ordinary Least Square Estimation of Gamma Autoregressive

Harmonic Model Parameters

$$Y_t = \phi Y_{t-1} + \beta \sin(\omega t) + \gamma \cos(\omega t) + \varepsilon_t \quad (3.50)$$

The residuals (ε_t) are given by:

$$\varepsilon_t = Y_t - (\phi Y_{t-1} + \beta \sin(\omega t) + \gamma \cos(\omega t)) \quad (3.51)$$

The sum of squared residuals (SSR) is:

$$SSR = \sum_{t=1}^n \varepsilon_t^2 = \sum_{t=1}^n (Y_t - \phi Y_{t-1} - \beta \sin(\omega t) - \gamma \cos(\omega t))^2 \quad (3.52)$$

$$\frac{\partial SSR}{\partial \phi} = -2 \sum_{t=1}^n Y_{t-1} (Y_t - \phi Y_{t-1} - \beta \sin(\omega t) - \gamma \cos(\omega t)) = 0 \quad (3.53)$$

Rearranging:

$$\sum_{t=1}^n Y_t Y_{t-1} = \phi \sum_{t=1}^n Y_{t-1}^2 + \beta \sum_{t=1}^n Y_{t-1} \sin(\omega t) + \gamma \sum_{t=1}^n Y_{t-1} \cos(\omega t) \quad (3.54)$$

$$\frac{\partial SSR}{\partial \beta} = -2 \sum_{t=1}^n \sin(\omega t) (Y_t - \phi Y_{t-1} - \beta \sin(\omega t) - \gamma \cos(\omega t)) = 0 \quad (3.55)$$

Rearranging:

$$\sum_{t=1}^n Y_t \sin(\omega t) = \phi \sum_{t=1}^n Y_{t-1} \sin(\omega t) + \beta \sum_{t=1}^n \sin^2(\omega t) + \gamma \sum_{t=1}^n \sin(\omega t) \cos(\omega t) \quad (3.56)$$

$$\frac{\partial SSR}{\partial \gamma} = -2 \sum_{t=1}^n \cos(\omega t) (Y_t - \phi Y_{t-1} - \beta \sin(\omega t) - \gamma \cos(\omega t)) = 0 \quad (3.57)$$

Rearranging:

$$\sum_{t=1}^n Y_t \cos(\omega t) = \phi \sum_{t=1}^n Y_{t-1} \cos(\omega t) + \beta \sum_{t=1}^n \sin(\omega t) \cos(\omega t) + \gamma \sum_{t=1}^n \cos^2(\omega t) \quad (3.58)$$

Solving for ϕ, β, γ

The three normal equations (3.54, 3.56 and 3.58) can be written in matrix form:

$$\begin{bmatrix} \sum Y_{t-1}^2 & \sum Y_{t-1} \sin(\omega t) & \sum Y_{t-1} \cos(\omega t) \\ \sum Y_{t-1} \sin(\omega t) & \sum \sin^2(\omega t) & \sum \sin(\omega t) \cos(\omega t) \\ \sum Y_{t-1} \cos(\omega t) & \sum \sin(\omega t) \cos(\omega t) & \sum \cos^2(\omega t) \end{bmatrix} \begin{bmatrix} \hat{\phi} \\ \hat{\beta} \\ \hat{\gamma} \end{bmatrix} = \begin{bmatrix} \sum Y_t Y_{t-1} \\ \sum Y_t \sin(\omega t) \\ \sum Y_t \cos(\omega t) \end{bmatrix} \quad (3.59)$$

Solving this system gives the OLS estimates for $\hat{\phi}, \hat{\beta}, \hat{\gamma}$.

3.5 Generalized Gamma Autoregressive Harmonic Model

For the extended model:

$$Y_t = \sum_{i=1}^p \Phi_i Y_{t-i} + \sum_{j=1}^q [\beta_j \sin(\omega_j t) + \gamma_j \cos(\omega_j t)] + \epsilon_t \quad (3.60)$$

where:

β and γ represent the coefficients for the sinusoidal components

p represents the order of the autoregressive part

q represents the number of harmonic components

ω_j represents different frequencies for each harmonic component

ϵ_t represents error term, typically assumed to follow gamma distribution.

3.5.1 Parameter Estimation of Generalized Gamma Autoregressive Harmonic Model

$$L(\theta) = \prod_{t=1}^n \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{\epsilon_t^2}{2\sigma^2}\right) \quad (3.61)$$

$$\begin{aligned} \ln L(\theta) = & -\frac{n}{2} \ln(2\pi) - \frac{n}{2} \ln(\sigma^2) \\ & - \frac{1}{2\sigma^2} \sum_{t=1}^n \left[Y_t - \left(\sum_{i=1}^p \Phi_i Y_{t-i} + \sum_{j=1}^q [\beta_j \sin(\omega_j t) + \gamma_j \cos(\omega_j t)] \right) \right]^2 \end{aligned} \quad (3.62)$$

$$\begin{aligned} \ln L(\theta) = & -\frac{n}{2} \ln(2\pi) - \frac{n}{2} \ln(\sigma^2) \\ & - \frac{1}{2\sigma^2} \sum_{t=1}^n \left[Y_t - \left(\sum_{i=1}^p \Phi_i Y_{t-i} + \sum_{j=1}^q [\beta_j \sin(\omega_j t) + \gamma_j \cos(\omega_j t)] \right) \right]^2 \end{aligned} \quad (3.63)$$

$$\frac{1}{2\sigma^2} \sum_{t=1}^n \left[Y_t - \left(\sum_{i=1}^p \Phi_i Y_{t-i} + \sum_{j=1}^q [\beta_j \sin(\omega_j t) + \gamma_j \cos(\omega_j t)] \right) \right]^2 \quad (3.64)$$

$$\epsilon_t = Y_t - \left(\sum_{i=1}^p \Phi_i Y_{t-i} + \sum_{j=1}^q [\beta_j \sin(\omega_j t) + \gamma_j \cos(\omega_j t)] \right) \quad (3.66)$$

$$\ln L(\theta) = -\frac{n}{2} \ln(2\pi) - \frac{n}{2} \ln(\sigma^2) - \frac{1}{2\sigma^2} \sum_{t=1}^n \epsilon_t^2 \quad (3.65)$$

Considering last term in (3.65) and differentiate $\ln L(\theta)$ with respect to Φ_i

$$\frac{\partial \ln L(\theta)}{\partial \Phi_i} = -\frac{1}{2\sigma^2} \cdot 2 \sum_{t=1}^n \epsilon_t \frac{\partial \epsilon_t}{\partial \Phi_i} \quad (3.66)$$

$$\frac{\partial \ln L(\theta)}{\partial \Phi_i} = -\frac{1}{\sigma^2} \sum_{t=1}^n \epsilon_t \frac{\partial \epsilon_t}{\partial \Phi_i} \quad (3.67)$$

$$\frac{\partial \ln L(\theta)}{\partial \Phi_i} = -\frac{1}{\sigma^2} \sum_{t=1}^n \epsilon_t (-Y_{t-i}) \quad (3.68)$$

$$\frac{\partial \ln L(\theta)}{\partial \Phi_i} = \frac{1}{\sigma^2} \sum_{t=1}^n \epsilon_t Y_{t-i} \quad (3.69)$$

$$\frac{\partial \ln L(\theta)}{\partial \Phi_i} = \frac{1}{\sigma^2} \sum_{t=1}^n \left[Y_t - \left(\sum_{i=1}^p \Phi_i Y_{t-i} + \sum_{j=1}^q [\beta_j \sin(\omega_j t) + \gamma_j \cos(\omega_j t)] \right) \right] Y_{t-i} = 0 \quad (3.70)$$

solve for Φ_i :

$$\frac{\partial \ln L(\theta)}{\partial \Phi_i} = \frac{1}{\sigma^2} \sum_{t=1}^n \left[Y_t - \left(\Phi_i Y_{t-i} + \sum_{j=1}^q [\beta_j \sin(\omega_j t) + \gamma_j \cos(\omega_j t)] \right) \right] Y_{t-i} = 0 \quad (3.71)$$

$$\sum_{t=1}^n \left[Y_t - \left(\Phi_i Y_{t-i} + \sum_{j=1}^q [\beta_j \sin(\omega_j t) + \gamma_j \cos(\omega_j t)] \right) \right] Y_{t-i} = 0 \quad (3.72)$$

$$\sum_{t=1}^n Y_t Y_{t-i} - \Phi_i \sum_{t=1}^n Y_{t-i}^2 - \sum_{t=1}^n \sum_{j=1}^q [\beta_j \sin(\omega_j t) + \gamma_j \cos(\omega_j t)] Y_{t-i} = 0 \quad (3.73)$$

$$\Phi_i \sum_{t=1}^n Y_{t-i}^2 = \sum_{t=1}^n Y_t Y_{t-i} - \sum_{t=1}^n \sum_{j=1}^q [\beta_j \sin(\omega_j t) + \gamma_j \cos(\omega_j t)] Y_{t-i} \quad (3.74)$$

$$\hat{\Phi}_i = \frac{\sum_{t=1}^n Y_t Y_{t-i} - \sum_{t=1}^n \sum_{j=1}^q [\beta_j \sin(\omega_j t) + \gamma_j \cos(\omega_j t)] Y_{t-i}}{\sum_{t=1}^n Y_{t-i}^2} \quad (3.75)$$

$$\ln L(\theta) = -\frac{1}{2\sigma^2} \sum_{t=1}^n \epsilon_t^2 \quad (3.76)$$

The derivative with respect to β_j gives:

$$\frac{\partial \ln L}{\partial \beta_j} = -\frac{1}{\sigma^2} \sum_{t=1}^n \epsilon_t \frac{\partial \epsilon_t}{\partial \beta_j} \quad (3.77)$$

$$\frac{\partial \ln L}{\partial \beta_j} = \frac{1}{\sigma^2} \sum_{t=1}^n \left[Y_t - \left(\sum_{i=1}^p \Phi_i Y_{t-i} + \sum_{j=1}^q [\beta_j \sin(\omega_j t) + \gamma_j \cos(\omega_j t)] \right) \right] \sin(\omega_j t) \quad (3.78)$$

$$\sum_{t=1}^n \left[Y_t - \left(\sum_{i=1}^p \Phi_i Y_{t-i} + \sum_{j=1}^q [\beta_j \sin(\omega_j t) + \gamma_j \cos(\omega_j t)] \right) \right] \sin(\omega_j t) = 0 \quad (3.79)$$

Now, solve for β_j . Expand the terms and isolate β_j :

$$\begin{aligned} \sum_{t=1}^n Y_t \sin(\omega_j t) - \sum_{t=1}^n \sum_{i=1}^p \Phi_i Y_{t-i} \sin(\omega_j t) - \sum_{t=1}^n \sum_{k=1}^q \beta_k \sin(\omega_k t) \sin(\omega_j t) \\ - \sum_{t=1}^n \gamma_k \cos(\omega_k t) \sin(\omega_j t) = 0 \end{aligned} \quad (3.80)$$

Finally, the estimate for β_j is:

$$\begin{aligned} \hat{\beta}_j \\ = \frac{\sum_{t=1}^n Y_t \sin(\omega_j t) - \sum_{t=1}^n \sum_{i=1}^p \Phi_i Y_{t-i} \sin(\omega_j t) - \sum_{t=1}^n \sum_{k=1}^q \gamma_k \cos(\omega_k t) \sin(\omega_j t)}{\sum_{t=1}^n \sin^2(\omega_j t)} \end{aligned} \quad (3.81)$$

$$\frac{\partial \ln L}{\partial \gamma_j} = -\frac{1}{\sigma^2} \sum_{t=1}^n \epsilon_t \frac{\partial \epsilon_t}{\partial \gamma_j} \quad (3.82)$$

$$\epsilon_t = Y_t - \left(\sum_{i=1}^p \Phi_i Y_{t-i} + \sum_{j=1}^q [\beta_j \sin(\omega_j t) + \gamma_j \cos(\omega_j t)] \right) \quad (3.83)$$

$$\frac{\partial \ln L}{\partial \gamma_j} = \frac{1}{\sigma^2} \sum_{t=1}^n \left[Y_t - \left(\sum_{i=1}^p \Phi_i Y_{t-i} + \sum_{j=1}^q [\beta_j \sin(\omega_j t) + \gamma_j \cos(\omega_j t)] \right) \right] \cos(\omega_j t) \quad (3.84)$$

$$\sum_{t=1}^n \left[Y_t - \left(\sum_{i=1}^p \Phi_i Y_{t-i} + \sum_{j=1}^q [\beta_j \sin(\omega_j t) + \gamma_j \cos(\omega_j t)] \right) \right] \cos(\omega_j t) = 0 \quad (3.85)$$

In (3.85), solve for γ_j . Expand the terms and isolate γ_j :

$$\begin{aligned}
& \sum_{t=1}^n Y_t \cos(\omega_j t) - \sum_{t=1}^n \sum_{i=1}^p \Phi_i Y_{t-i} \cos(\omega_j t) - \sum_{t=1}^n \sum_{k=1}^q \beta_k \sin(\omega_k t) \cos(\omega_j t) \\
& - \sum_{t=1}^n \gamma_k \cos(\omega_k t) \cos(\omega_j t) = 0
\end{aligned} \tag{3.86}$$

$$\begin{aligned}
& \hat{\gamma}_j \\
& = \frac{\sum_{t=1}^n Y_t \cos(\omega_j t) - \sum_{t=1}^n \sum_{i=1}^p \Phi_i Y_{t-i} \cos(\omega_j t) - \sum_{t=1}^n \sum_{k=1}^q \beta_k \sin(\omega_k t) \cos(\omega_j t)}{\sum_{t=1}^n \cos^2(\omega_j t)}
\end{aligned} \tag{3.87}$$

3.5.2 Ordinary Least Square Estimation of Generalized Gamma

Autoregressive Harmonic Model Parameters

$$Y_t = \sum_{i=1}^p \Phi_i Y_{t-i} + \sum_{j=1}^q [\beta_j \sin(\omega_j t) + \gamma_j \cos(\omega_j t)] + \epsilon_t \tag{3.88}$$

The residuals (ϵ_t) are:

$$\epsilon_t = Y_t - \sum_{i=1}^p \Phi_i Y_{t-i} - \sum_{j=1}^q [\beta_j \sin(\omega_j t) + \gamma_j \cos(\omega_j t)] \tag{3.89}$$

$$SSR = \sum_{t=1}^n \epsilon_t^2 = \sum_{t=1}^n \left(Y_t - \sum_{i=1}^p \Phi_i Y_{t-i} - \sum_{j=1}^q [\beta_j \sin(\omega_j t) + \gamma_j \cos(\omega_j t)] \right)^2 \tag{3.90}$$

Minimizing SSR with respect to each parameter gives the OLS estimates.

For Φ_i (for $i = 1, \dots, p$)

$$\frac{\partial SSR}{\partial \Phi_i} = -2 \sum_{t=1}^n Y_{t-i} \left(Y_t - \sum_{i=1}^p \Phi_i Y_{t-i} - \sum_{j=1}^q [\beta_j \sin(\omega_j t) + \gamma_j \cos(\omega_j t)] \right) = 0 \tag{3.91}$$

Rearranging:

$$\sum_{t=1}^n Y_t Y_{t-i} = \sum_{i=1}^p \Phi_i \sum_{t=1}^n Y_{t-i}^2 + \sum_{j=1}^q \beta_j \sum_{t=1}^n Y_{t-i} \sin(\omega_j t) + \sum_{j=1}^q \gamma_j \sum_{t=1}^n Y_{t-i} \cos(\omega_j t) \quad (3.92)$$

For β_j (for $j = 1, \dots, q$)

$$\frac{\partial SSR}{\partial \beta_j} = -2 \sum_{t=1}^n \sin(\omega_j t) \left(Y_t - \sum_{i=1}^p \Phi_i Y_{t-i} - \sum_{j=1}^q [\beta_j \sin(\omega_j t) + \gamma_j \cos(\omega_j t)] \right) = 0 \quad (3.93)$$

Rearranging:

$$\begin{aligned} \sum_{t=1}^n Y_t \sin(\omega_j t) &= \sum_{i=1}^p \Phi_i \sum_{t=1}^n Y_{t-i} \sin(\omega_j t) + \sum_{j=1}^q \beta_j \sum_{t=1}^n \sin^2(\omega_j t) \\ &\quad + \sum_{j=1}^q \gamma_j \sum_{t=1}^n \sin(\omega_j t) \cos(\omega_j t) \end{aligned} \quad (3.94)$$

For γ_j (for $j = 1, \dots, q$)

$$\frac{\partial SSR}{\partial \gamma_j} = -2 \sum_{t=1}^n \cos(\omega_j t) \left(Y_t - \sum_{i=1}^p \Phi_i Y_{t-i} - \sum_{j=1}^q [\beta_j \sin(\omega_j t) + \gamma_j \cos(\omega_j t)] \right) = 0 \quad (3.95)$$

Rearranging:

$$\begin{aligned} \sum_{t=1}^n Y_t \cos(\omega_j t) &= \sum_{i=1}^p \Phi_i \sum_{t=1}^n Y_{t-i} \cos(\omega_j t) + \sum_{j=1}^q \beta_j \sum_{t=1}^n \sin(\omega_j t) \cos(\omega_j t) \\ &\quad + \sum_{j=1}^q \gamma_j \sum_{t=1}^n \cos^2(\omega_j t) \end{aligned} \quad (3.96)$$

The system of normal equations can be written in matrix form:

$$\begin{bmatrix} \sum Y_{t-1}^2 & \sum Y_{t-1}Y_{t-2} & \cdots & \sum Y_{t-1} \sin(\omega_1 t) & \sum Y_{t-1} \cos(\omega_1 t) & \cdots \\ \sum Y_{t-2}Y_{t-1} & \sum Y_{t-2}^2 & \cdots & \sum Y_{t-2} \sin(\omega_1 t) & \sum Y_{t-2} \cos(\omega_1 t) & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots \\ \sum Y_{t-1} \sin(\omega_1 t) & \sum Y_{t-2} \sin(\omega_1 t) & \cdots & \sum \sin^2(\omega_1 t) & \sum \sin(\omega_1 t) \cos(\omega_1 t) & \cdots \\ \sum Y_{t-1} \cos(\omega_1 t) & \sum Y_{t-2} \cos(\omega_1 t) & \cdots & \sum \sin(\omega_1 t) \cos(\omega_1 t) & \sum \cos^2(\omega_1 t) & \cdots \\ \vdots & \vdots & \cdots & \vdots & \vdots & \ddots \end{bmatrix}
\begin{bmatrix} \hat{\Phi}_1 \\ \hat{\Phi}_2 \\ \vdots \\ \hat{\beta}_1 \\ \hat{\gamma}_1 \\ \vdots \end{bmatrix} = \begin{bmatrix} \sum Y_t Y_{t-1} \\ \sum Y_t Y_{t-2} \\ \vdots \\ \sum Y_t \sin(\omega_1 t) \\ \sum Y_t \cos(\omega_1 t) \\ \vdots \end{bmatrix} \quad (3.97)$$

Solving this system provides the OLS estimates for $\Phi_i, \beta_j, \gamma_j$.

3.6 Optimal Models Selection Criteria

A variety of factors are taken into consideration while choosing the best model. They include the Hanan-Quinn criterion, Bayesian estimate, Schwarz, Rissanen, and Akaike's information, among others. In 1969, Akaike and Akaike introduced the ultimate prediction error. In 1977, Söderström expanded on Akaike's (1969) and Olatayo's (2013) analysis of the AR(p) order to incorporate ARMA (p,q) models.

The minimized criterion is given as

$$PPE = (\sigma^2)^{\frac{n+p+q}{n-p-q}} \quad (3.98)$$

Around 1970, Akaike noticed that FPE gets steadily incoherent. In 1973, he developed his information criteria by drawing on informational-theoretic considerations. This statistic was designed to provide a consistently unbiased assessment of Kullback-Leibler indices for the fitted model in comparison to the original form, according to Akaike (1973).

$$\text{AIC} = -2 \ln \text{Likelihood}(\hat{\phi}, \hat{\theta}, \hat{\sigma}^2) + 2(p + q + 1) \quad (3.99)$$

Using a metric, such AIC, that limits the likelihood of a specific number of model components, the most economical model is selected. In 1978, Akaike created the Bayesian information criterion (BIC), an ordered selecting criterion, by modifying AIC using a Bayesian approach (Akaike, 1978).

Bayesian information criteria (BIC) is given as;

$$\text{BIC} = (n - p - q) \ln \left(\frac{n\hat{\sigma}^2}{n-p-q} \right) + n(1 + \ln \sqrt{2\pi}) + (p + q) \ln \left[\frac{\sum_{t=1}^n (x_t^2 - n\hat{\sigma}^2)}{p+q} \right] \quad (3.100)$$

There is sufficient evidence to suggest BIC offers a better ARMA selection of models' criterion than AIC, given AIC's tendency to select over-parameterized models (Olatayo and Taiwo, 2014).

Using Bayesian techniques and a preferred data-coding scheme, Schwarz (1978) and Rissanen (1978) independently developed the same criterion, which was later known as the Schwarz-Rissanen criterion.

$$\text{SIC} = \ln \hat{\sigma}^2 + \frac{(p+q)}{n} \ln n \quad (3.101)$$

Geweke and Mease (1981) proposed using the Bayesian estimation criteria, or BEC, to approximate SIC.

$$\text{BEC} = \hat{\sigma}^2 + (p_x + q_x) \hat{\sigma}_x^2 \ln \left(\frac{n}{n-p_x-q_x} \right) \quad (3.102)$$

Hannan and Quinn (1979) developed the Hannan-Quinn criterion (HQ) using the iterated logarithm's underlying rules. When compared to an estimator that is adequately stable, it provides an unfavorable function that decreases as fast as is practical as the sample size grows.

$$\text{HQ} = \ln \hat{\sigma}^2 + 2(p + q) \frac{\ln(\ln n)}{n} \quad (3.103)$$

3.7 Diagnostic Checking

Finding evidence of a pattern for residuals that resembles the most recent unobservable white noise sequence is the standard method. Next, it is verified which statistical features of this residual series actually mirror those of white noise. The majority of diagnostic processes focus on looking at residual criteria to identify whether white noise from the estimated model will be normally distributed, autonomous, and have stable variance. One of the first methods that can be used to assess if a model is sufficient is the visual depiction of error over time, which can be expressed as a graph (Shumay and Stoffer, 2010).

$$\omega_t = \frac{\chi_t - \chi_t^{(t-1)}}{\sqrt{\rho_t^{(t-1)}}} \quad (3.104)$$

$\chi_t - \chi_t^{t-1}$ is the time-ahead prediction of (χ_t) , and ρ_t^{t-1} is the predicted time-ahead error variance.

To determine whether or not the residual of the fitted model is white noise, one can also utilize the Autocorrelations Function (ACF) of the series. cc

The correlogram is determine as:

$$\gamma_{ak} = \frac{\sum_{t=k+1}^n \omega_t \omega_{t-k}}{\sum_{t=1}^n \omega_t^2} \quad (3.105)$$

If more than 5% of the coefficient deviates from this range, it is highly probable that the residual does not follow a white noise process (Lehmann and Rode, 2001).

3.7.1 Ljung-Box Q (LBQ) Statistic

To determine whether the model is sufficient, a hypothesis test may be conducted by choosing a level of significance and contrasting the resulting χ^2 value with the critical value from the χ^2 -table.

Ljung and Box (1978) test statistic is given as:

$$Q(r) = n'(n' + 2) \sum_{j=1}^k \frac{\mu_j^2}{n-j} \quad (3.106)$$

3.8 Forecasting Accuracy Measures

There are several ways to gauge how accurate forecasts and forecast accuracy measures are as follows (Olatayo and Taiwo, 2016). The MAE is provided as:

$$\frac{1}{h+1} \sum_{t=s}^{h+s} (\hat{X}_t + X_t)^2 \quad (3.107)$$

RMSE is defined as:

$$\text{RMSE} = \sqrt{\frac{1}{h+1} \sum_{t=s}^{h+s} (\hat{X}_t - X_t)^2} \quad (3.108)$$

and MAPE is given as

$$\text{MAPE} = \frac{100}{h+s} \sum_{t=s}^{h+s} \left| \frac{\hat{X}_t - X_t}{X_t} \right| \quad (3.109)$$

Where $t = s, 1 + s, \dots, h + s$.

CHAPTER FOUR

RESULTS AND DISCUSSION

The section is utilized to discuss the efficiency and performance of the Gamma Auto Regressive Harmonic Model (GARHM) using Nigeria's monthly exchange rate from January 1996 to February 2025, obtained from the Investing website; Nigeria's yearly data on temperature from 1901 to 2023, which was obtained from the World Bank; Nigeria's yearly data on CO₂ emission from 1990 to 2020, which was obtained from Macrotrend; and Nigeria's yearly data on macroeconomic variable (inflation) from 1961 to 2023, which was obtained from the World Bank.

4.1 Exploratory Data Analysis on the Datasets

The descriptive statistics in Table 4.1 and Figure 4.5 highlight the exchange rate dataset's nonstationary (ADF test statistic=1.9141, p-value = 0.613) and Non-Gaussian features. The series' mean value is 253.02, with a minimum of 75.72 and a maximum of 1670.89. The variance is particularly high (84598.23) with a positive skewness (3.4885).

The inflation rate dataset consists of 63 observations, with a mean value of 17.7519. The minimum and maximum inflation rate values recorded are -3.095 and 219.0028, respectively. The variance (865.1477) and standard deviation (29.413339) suggest a significant fluctuation over time. The skewness (5.3525) indicates occasional periods of high inflation spikes. The kurtosis confirms a leptokurtic distribution, meaning extreme inflation values occur more frequently than in a normal distribution.

The temperature dataset consists of 123 observations with the variance (0.1364) and standard deviation (0.3694) indicating low fluctuations in temperature over time. The skewness value

(0.2860) suggests asymmetric distribution. There is non-stationarity in the series, indicating a potential trend or seasonal pattern in temperature variations.

The CO₂ dataset consists of 31 observations, with a mean value of 95836.97. The minimum and maximum values recorded are 72768.8 and 119544.1, respectively, with a variance of 1.46630950 and a standard deviation of 12109.1. The skewness (0.09504) suggests a moderately right-skewed distribution, meaning that there are signs of extreme CO₂ emission. The kurtosis (2.1793) indicates a slightly platykurtic distribution, implying fewer extreme values than in a normal distribution. The ADF test statistic (0.693) confirms the presence of a unit root, signifying nonstationarity in CO₂ emissions over time.

A visual inspection of the Nigerian exchange rate time plot in Figure 4.1, covering 350 observations from January 1996 to February 2025, signified a nonstationary series. In the time plot, oscillations and dramatic upward changes demonstrate the series' strong trend, seasonal, and cyclical variations. The aforementioned attributes suggest that the exchange rate is heavily influenced by external economic forces and policy changes.

A visual inspection of the inflation rate time plot in Figure 4.2 from 1961 to 2023 indicates that there is a relatively low and stable inflation rate before 1970. Inflation appears more stable from 1998 compared to the relatively high periods but still fluctuates. All these indicated that the Nigerian inflation rate exhibits seasonal movement.

The temperature time series in Figure 4.3, which covers the period from 1901 to 2023, appears to show both long-term variations and an overall upward trend. There are seasonal fluctuations evident in the plot, which shows several intervals of notable rises and falls. Interestingly, notable

peaks appear in the early 21st century and around the middle of the 20th century, indicating times of increased variability. A progressive upward change in the annual mean temperature values is implied by the growing trend in recent decades, which could be caused by underlying structural, economic, or climatic causes.

Visual examination of the CO₂ emissions time series from 1990 to 2025 in Figure 4.4 indicates that the series shows an upward trend with sporadic oscillations. Emissions gradually increased in the early years, with a brief decrease in between, suggesting seasonal fluctuations. The plot's most striking feature is the dramatic increase in emissions after 2010, which has continued to be high in recent years. This points to possible external factors impacting CO₂ emissions, such as economic development, regulatory changes, or industrial expansion. The oscillating structure in the first portion of the record suggests short-term shocks or interventions, whereas the significant increasing trajectory after 2010 indicates a structural shift in emission levels.

Figures 4.9 to 4.16 exhibit a pattern where autocorrelations decay exponentially, further reinforcing the presence of non-stationarity. Given these properties, the developed models in this study are deemed appropriate for capturing the behaviour of Non-Gaussian observations and effectively modelling the datasets' dynamics.

Table 4.1: Descriptive Statistics of the Datasets

Dataset	N	Mean	Min.	Max.	Varianc	Std.Dev	Skewn	Kurtosis	ADF	P-value	J-Bera
s					e		ess				
Exchan	350	253.0219	75.72	1670.89	84598.23	290.8577	3.4885	12.6430	1.9141	0.613	2.2e-16
ge Rate											
Inflatio	63	17.75149	-3.094	219.0028	865.1477	29.41339	5.3525	36.25828	3.7821	0.0251	2.2e-16
n Rate											
Temper	123	26.99293	26.18	27.92	0.1364	0.3694	0.2860	2.4869	2.1225	0.5257	0.00522
ature											
CO ₂	31	95836.97	72768.8	119544.1	1.47e+08	12109.1	0.0950	2.1793	1.6878	0.693	0.00532

Source: Researcher’s Computation, 2025

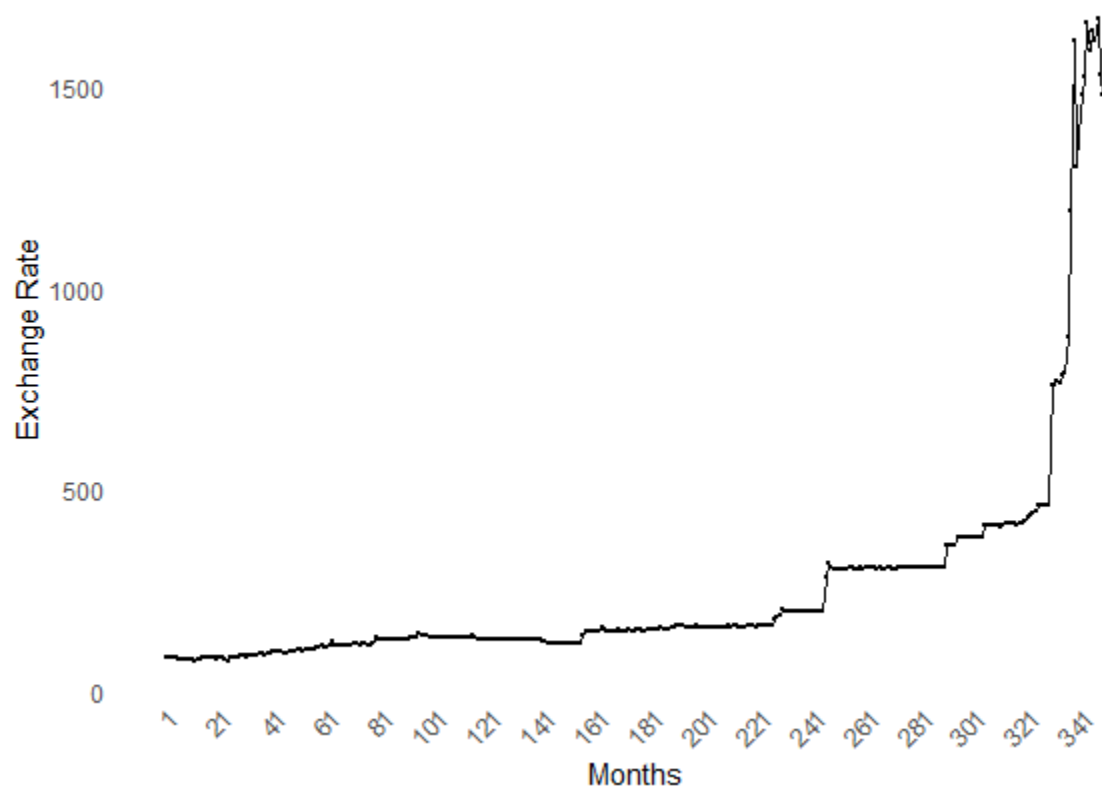


Figure 4.1: Time plot of Exchange rate

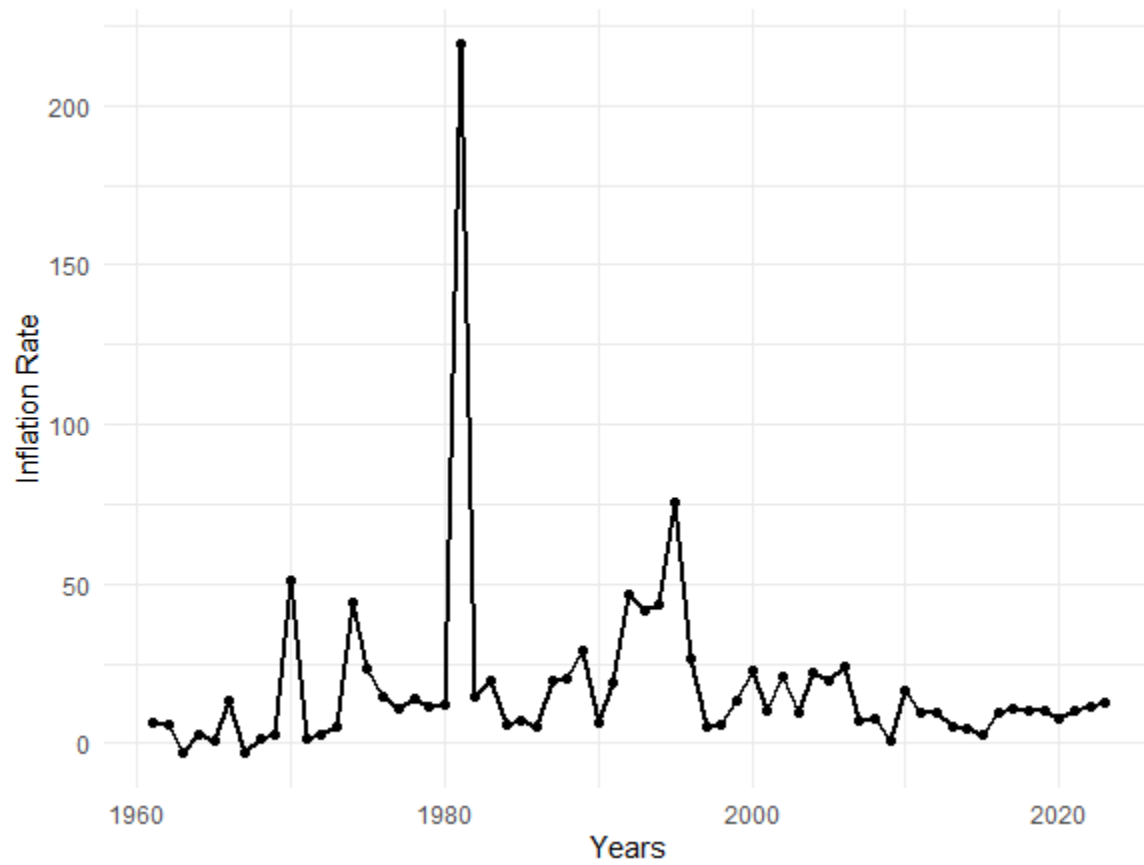


Figure 4.2: Time plot of Inflation rate

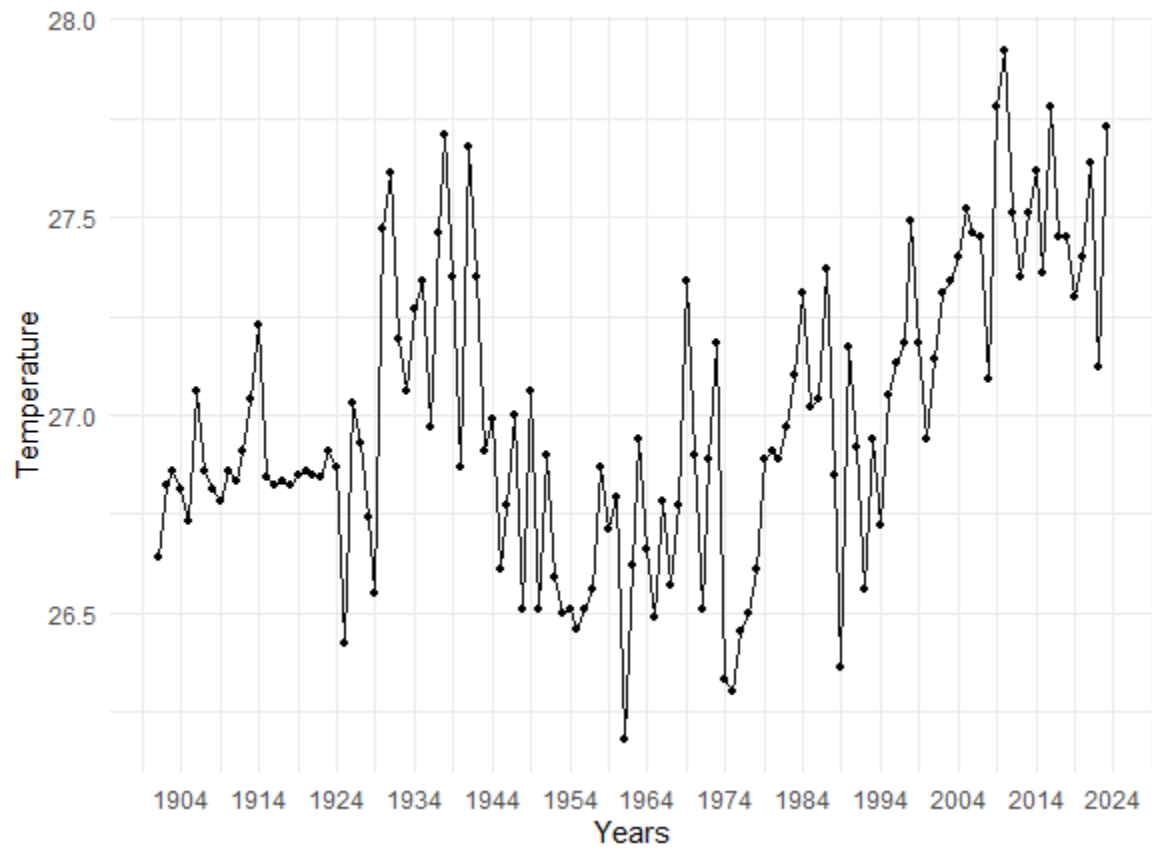


Figure 4.3: Time plot of Temperature

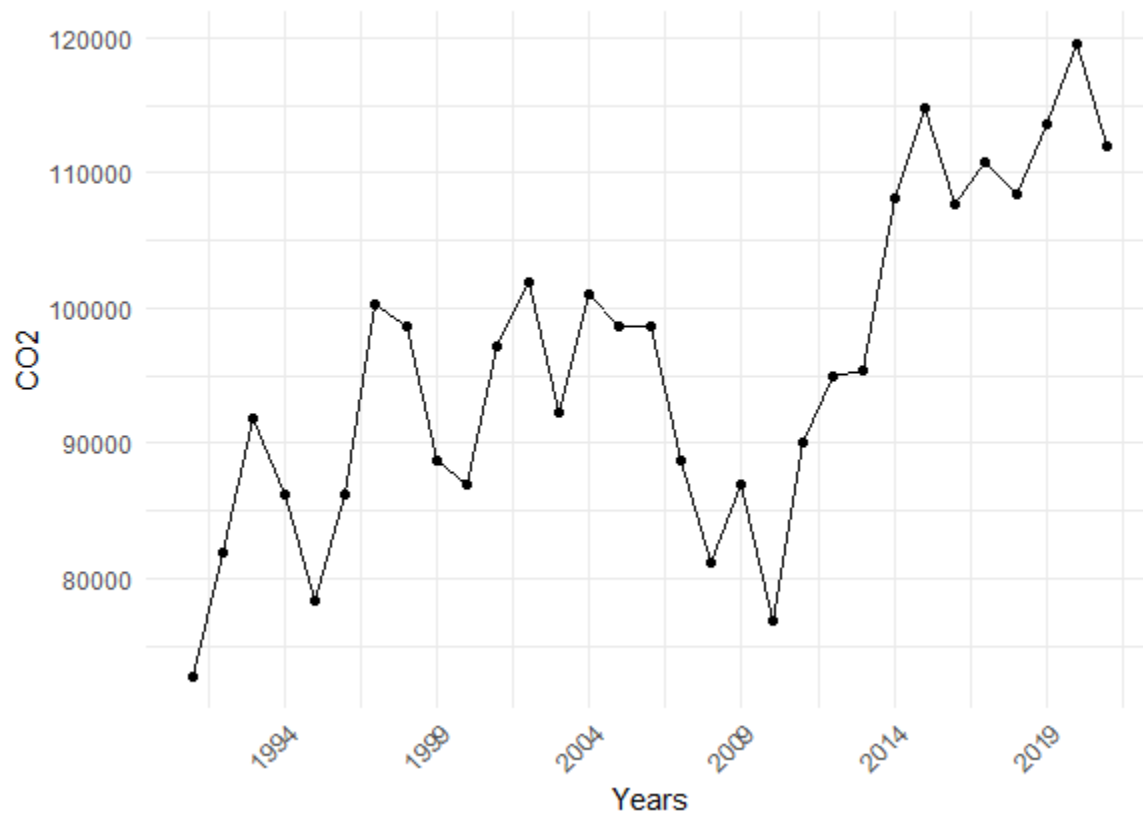


Figure 4.4: Time plot of CO₂ emission

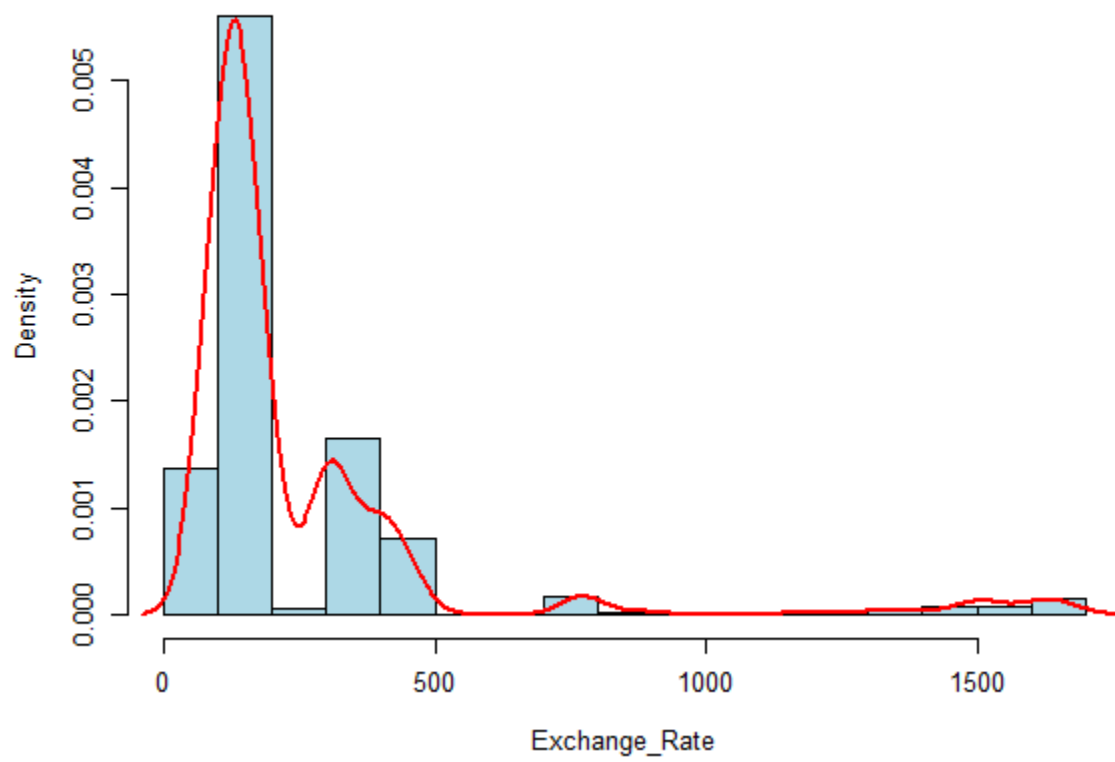


Figure 4.5: Histogram of Exchange rate

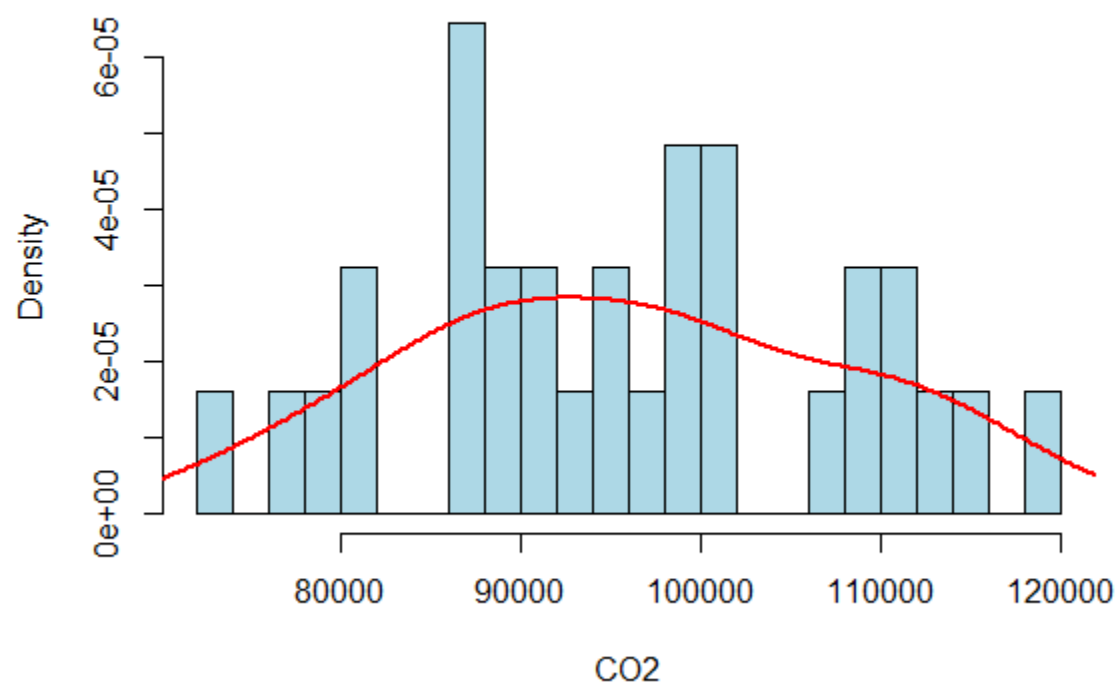


Figure 4.6: Histogram of CO₂

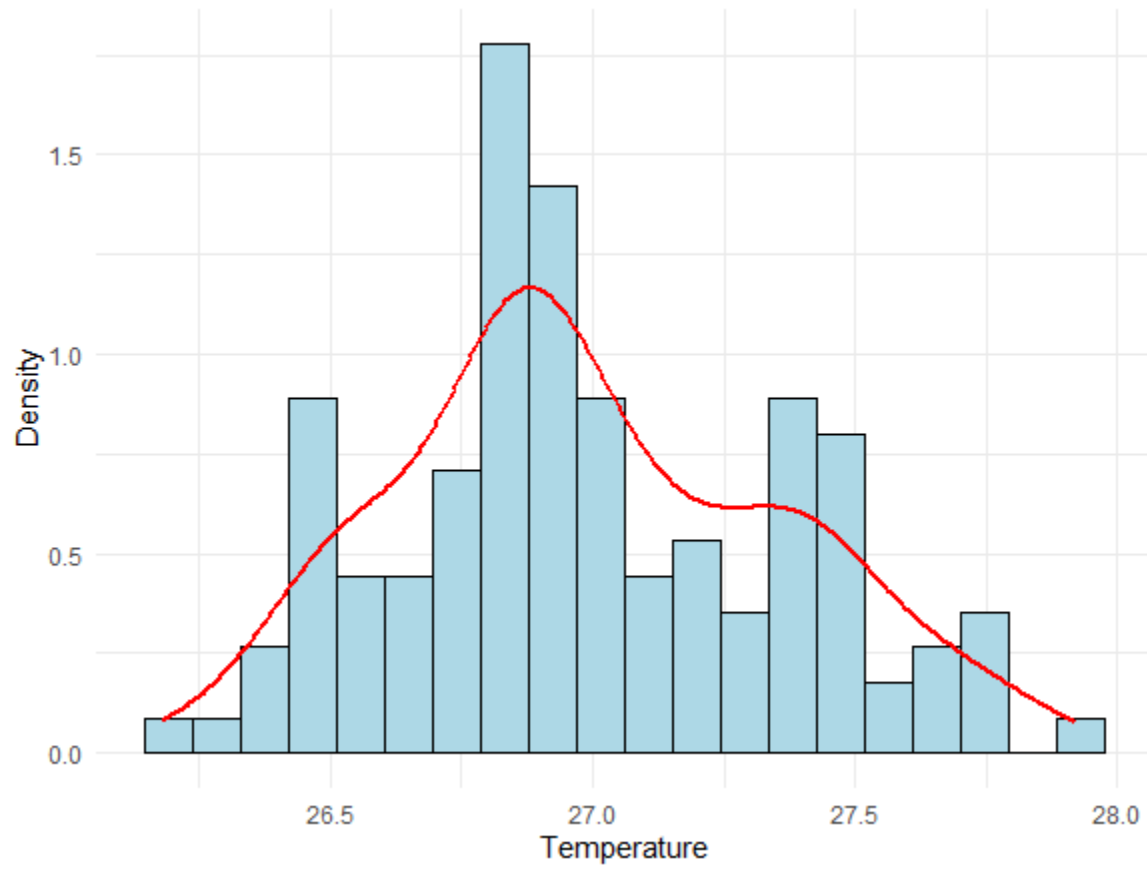


Figure 4.7: Histogram of Temperature

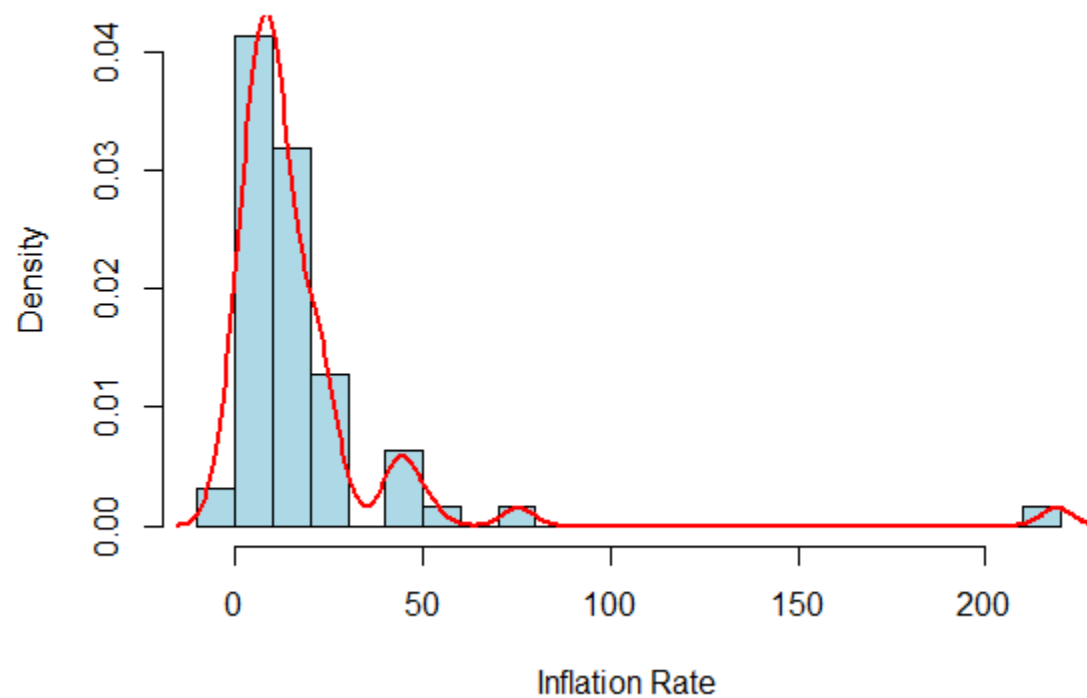


Figure 4.8: Histogram of Inflation Rate

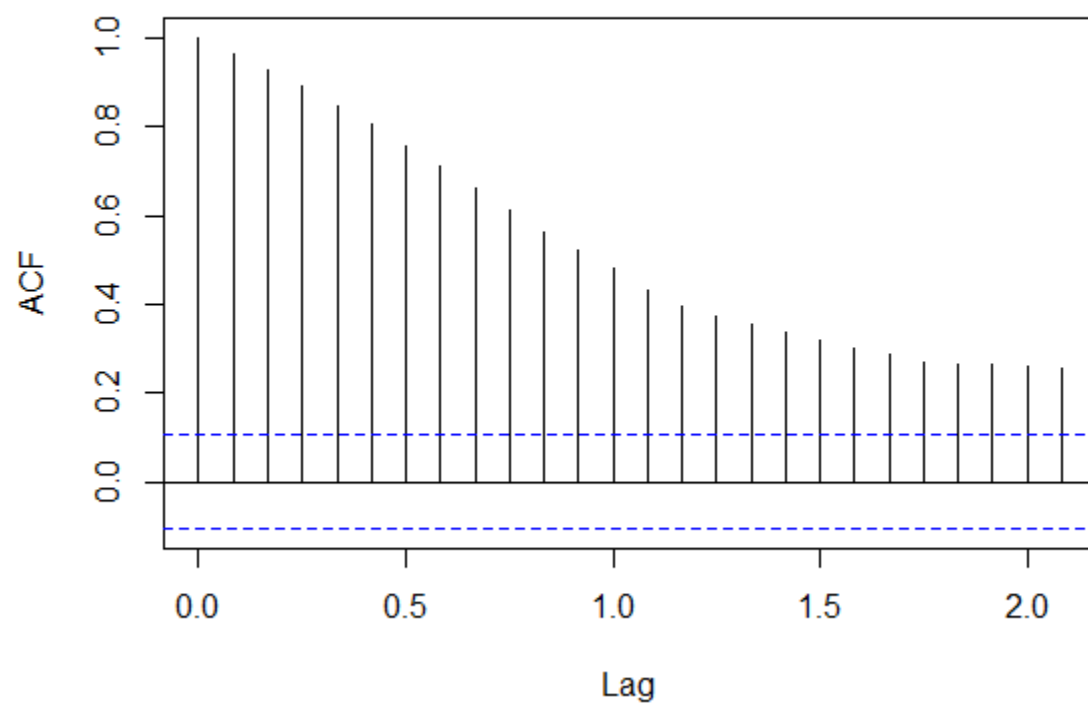


Figure 4.9: ACF of Exchange rate

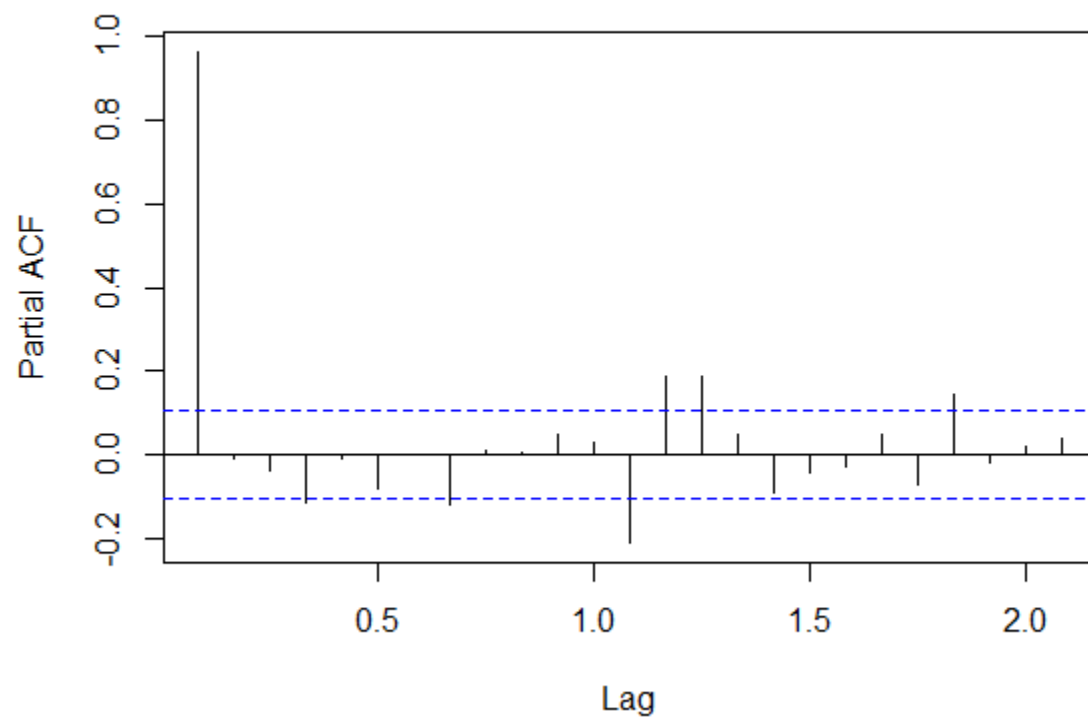


Figure 4.10: PACF of Exchange rate

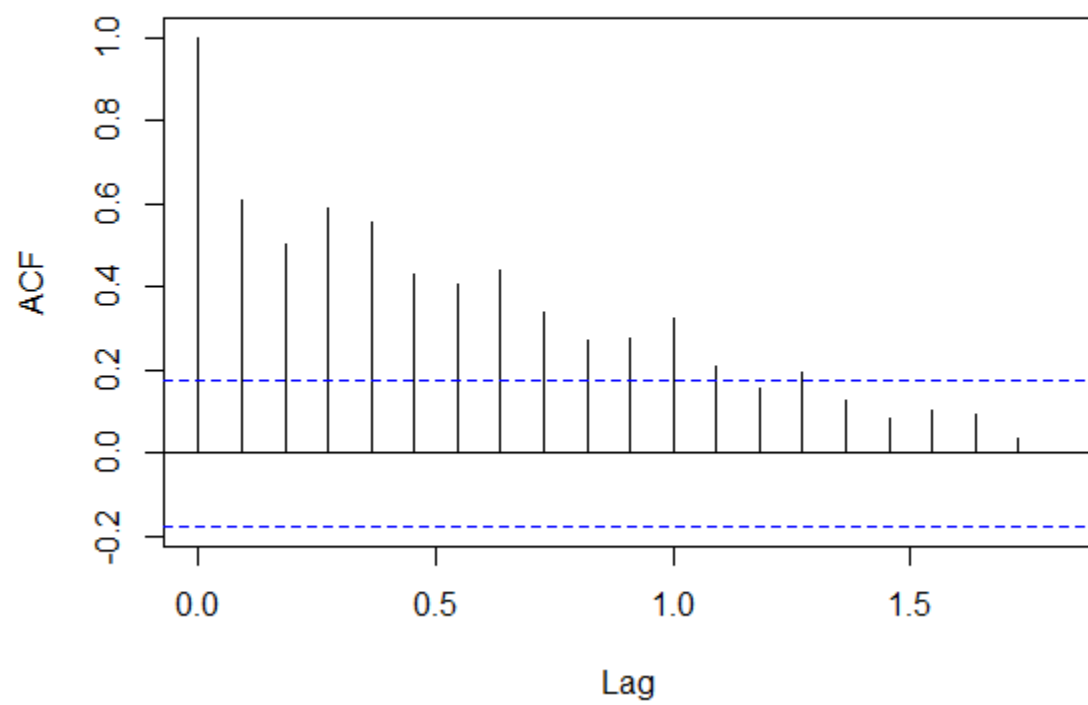


Figure 4.11: ACF of CO₂ emission

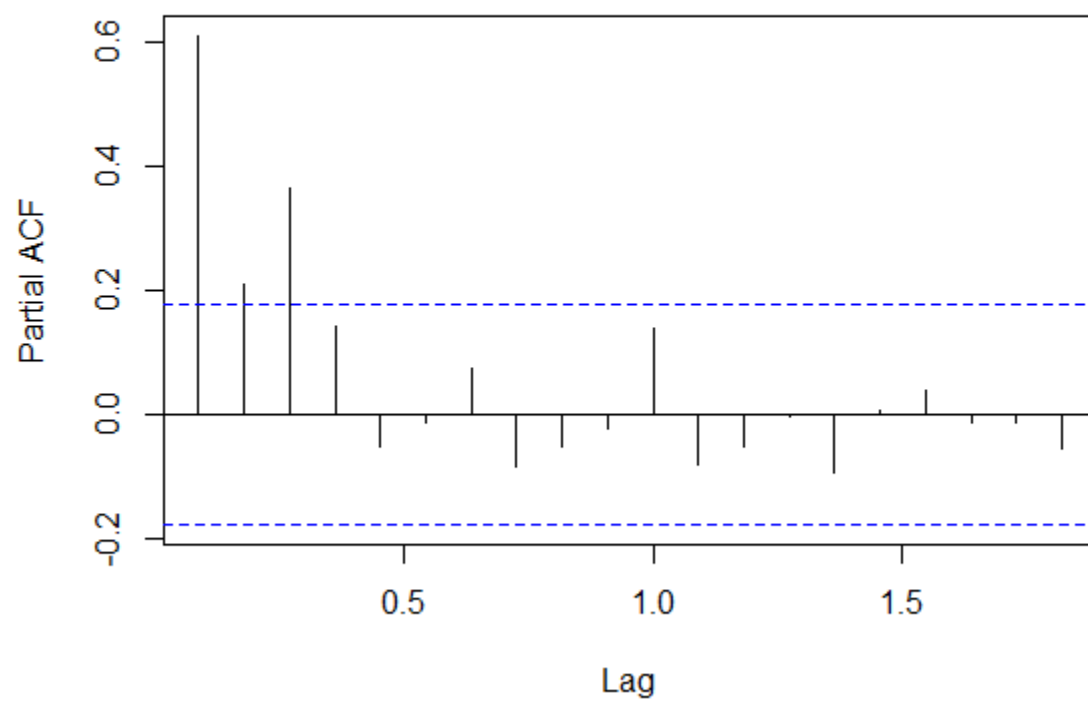


Figure 4.12: PACF of CO₂ emission

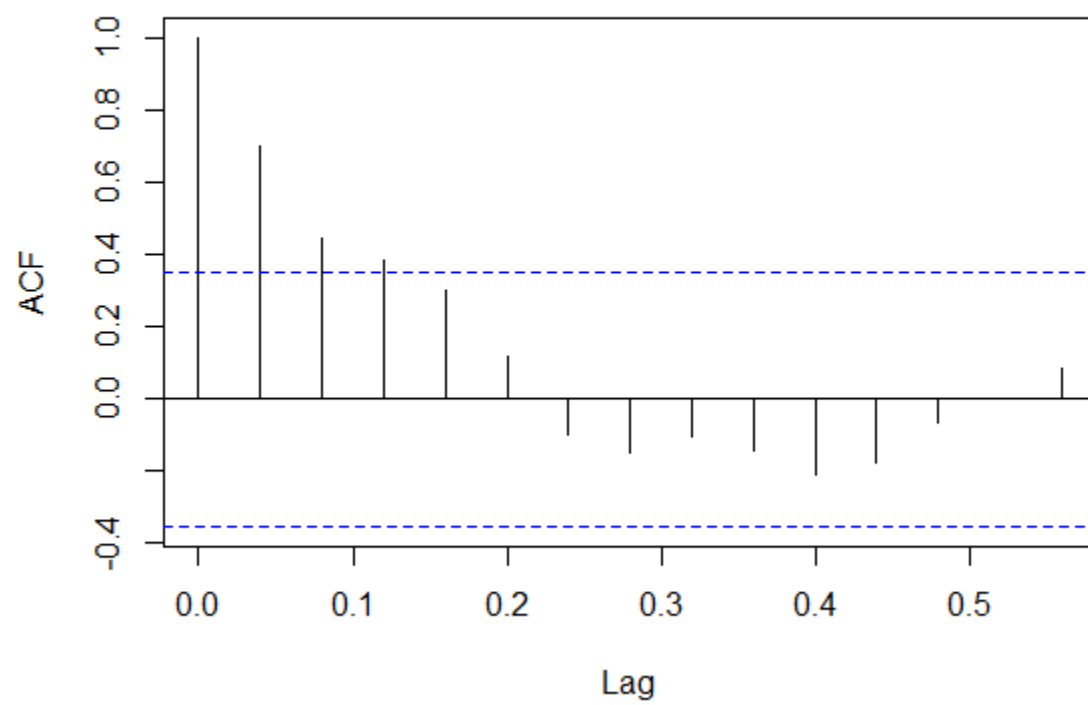


Figure 4.13: ACF of Temperature

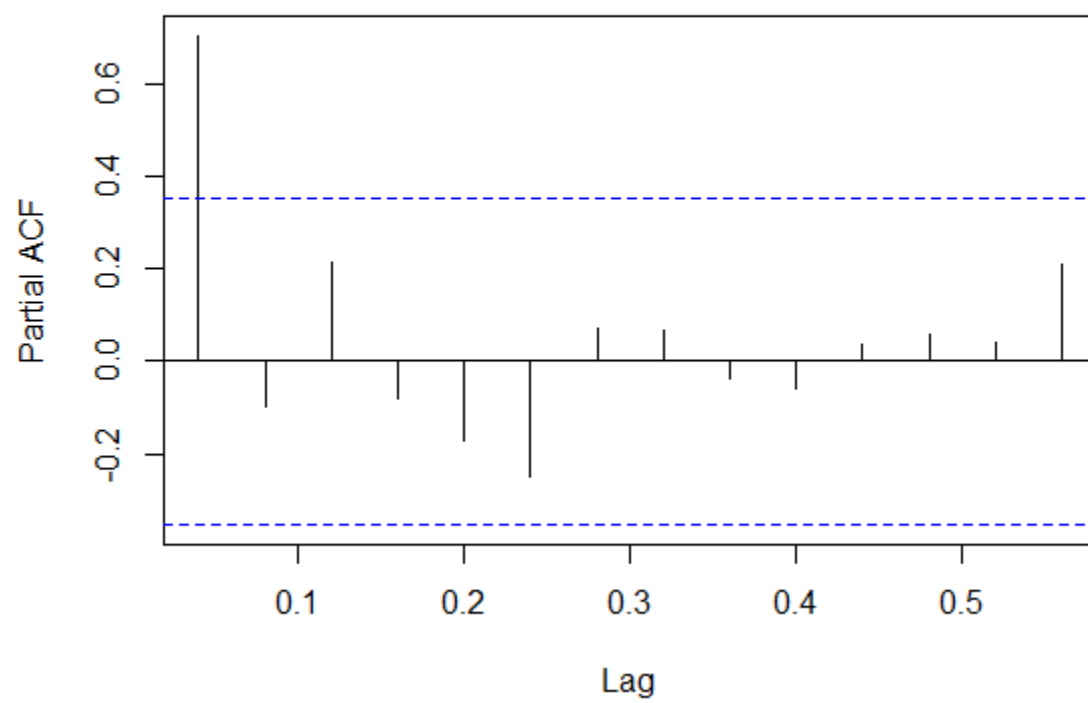


Figure 4.14: PACF of Temperature

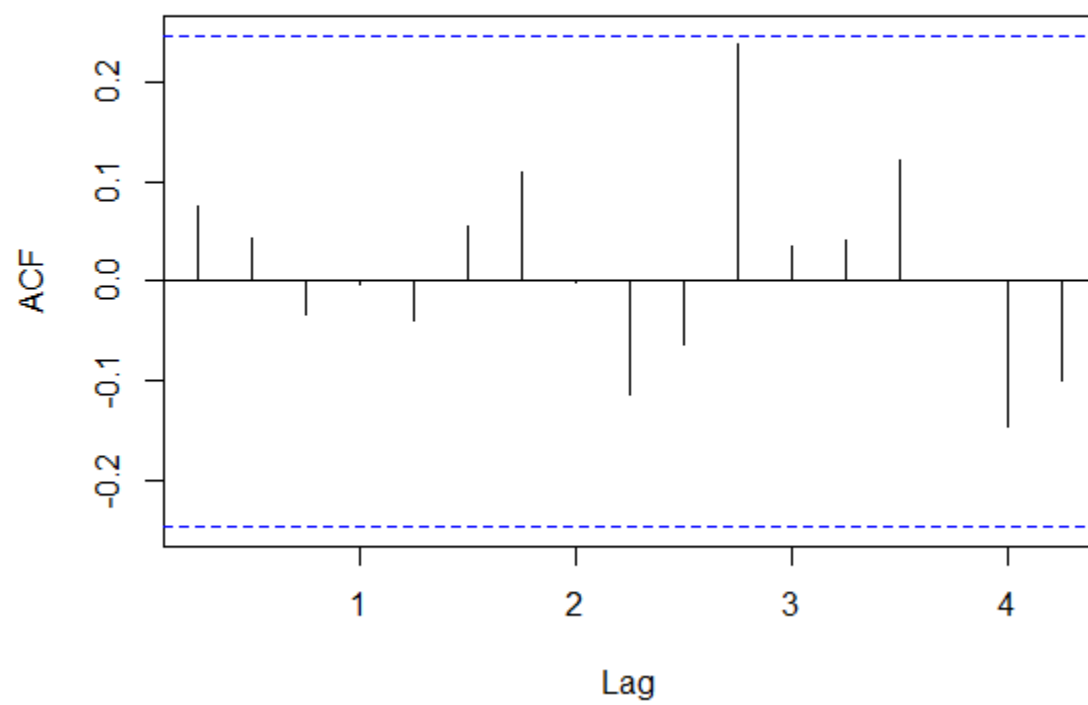


Figure 4.15: ACF of Inflation rate

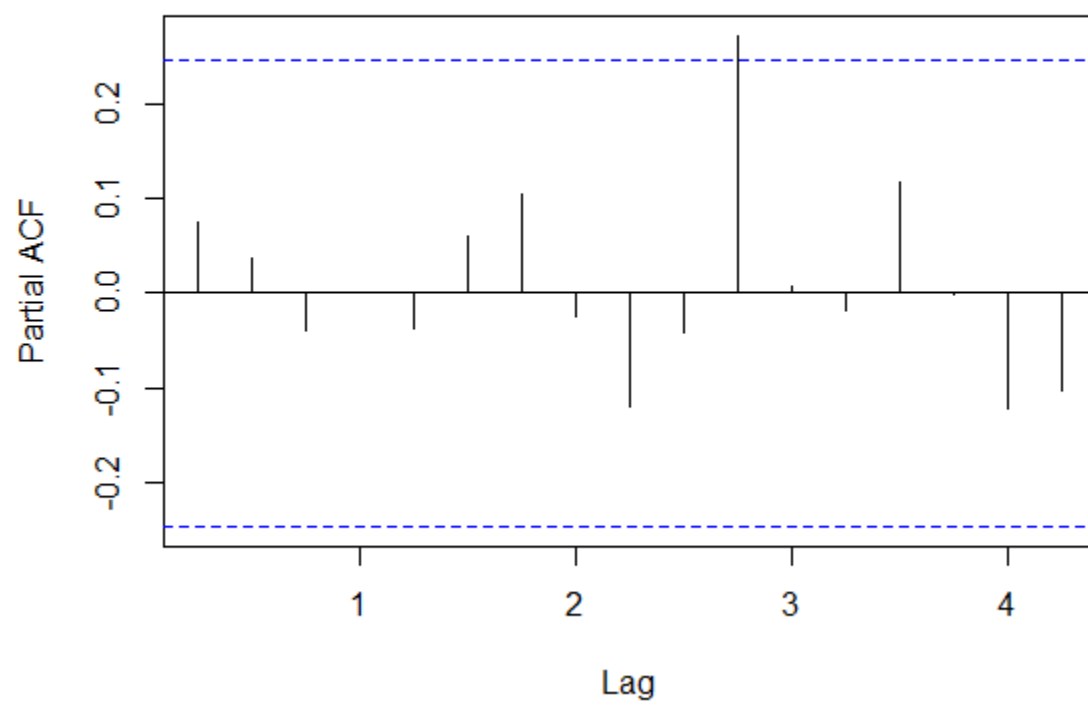


Figure 4.16: PACF of Inflation rate

4.2 Gamma Autoregressive Harmonic Model

4.2.1 Gamma Autoregressive Harmonic Model for Exchange Rate

The order of the developed model was shown in Table 4.2. The lowest AIC and BIC values indicate that the optimal choice is that of order 1 across all the models. To determine the dominant periodicity in the dataset, we applied spectral analysis techniques to identify the most significant frequency component. Specifically, we computed the periodogram of the time series and extracted the frequency corresponding to the peak spectral density. The dominant frequency was then used to compute the dominant period T using the relationship:

$$T = \frac{1}{f}$$

where T represents the dominant cycle length in the observed time series. The analysis revealed a dominant period of approximately 12, suggesting a strong seasonal component within this time frame. This periodicity was subsequently incorporated into the hybrid Gamma Autoregressive Harmonic Model in (4.1) to better capture both autoregressive dynamics and underlying oscillatory behavior.

The Gamma Autoregressive Harmonic Model estimated for the exchange rate is given as:

$$y_t = 370.7247296 + 0.9980131y_{t-1} + 328.689687\sin 0.167\pi_1 t + 173.9511088\cos 0.167\pi_1 t + \varepsilon_t \quad (4.1)$$

with $R^2 = 0.982$, *Adjusted* $R^2 = 0.97$ and Durbin Watson = 2.019162

Table 4.2: Order selection based on AIC and BIC values for Exchange rate

Model	AIC	BIC
AR (1)	3586.641	3598.215
AR (2)	3588.617	3604.048
AR (3)	3590.082	3609.371
Harmonic (1)	4970.936	4986.368
Harmonic (2)	4974.686	4997.834
Harmonic (3)	4978.585	5009.448
GARHM (1)	3585.903	3605.193
GARHM (2)	3588.394	3619.257
GARHM (3)	3588.298	3630.735

Source: Researcher's Computation, 2025

4.2.2 Gamma Autoregressive Harmonic Model Validation for Exchange Rate

The R^2 in Table 4.3 revealed that time explained the variations in exchange rate as 98.1% with no serial correlation (Equation 4.1). Additionally, the model $\bar{R}^2$ value showed that the model has a 97.1% predictive power.

The residual does not have the usual structure and is significant at $\alpha = 0.05$ with non-normality (Figures 4.17-4.19). Considering Durbin-Watson statistics value, Figures 4.17, 4.18, and 4.19, the Gamma Autoregressive Harmonic Model is considered adequate for the exchange rate dataset.

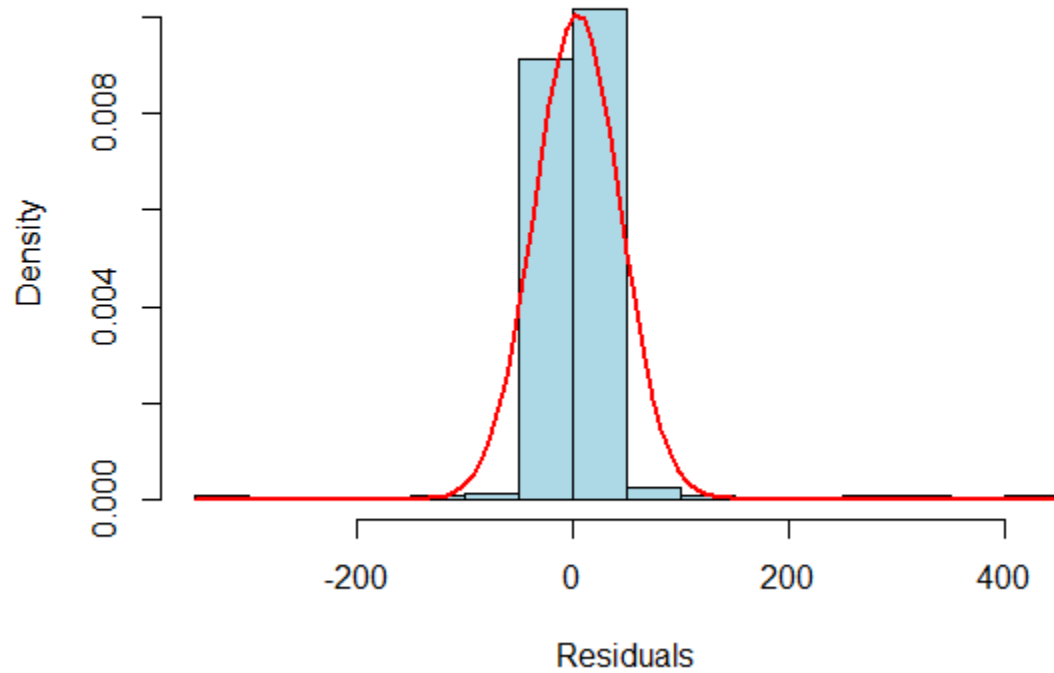


Figure 4.17: Histogram of the Gamma Autoregressive Harmonic Model for Exchange rate

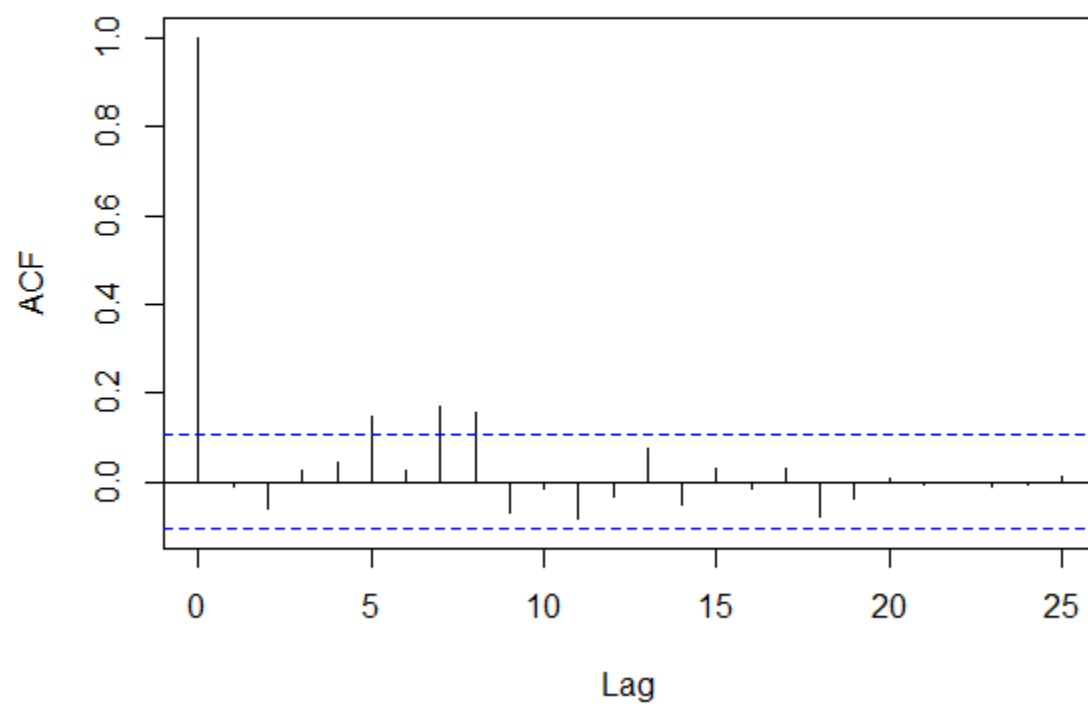


Figure 4.18: ACF of the Gamma Autoregressive Harmonic Model for Exchange rate

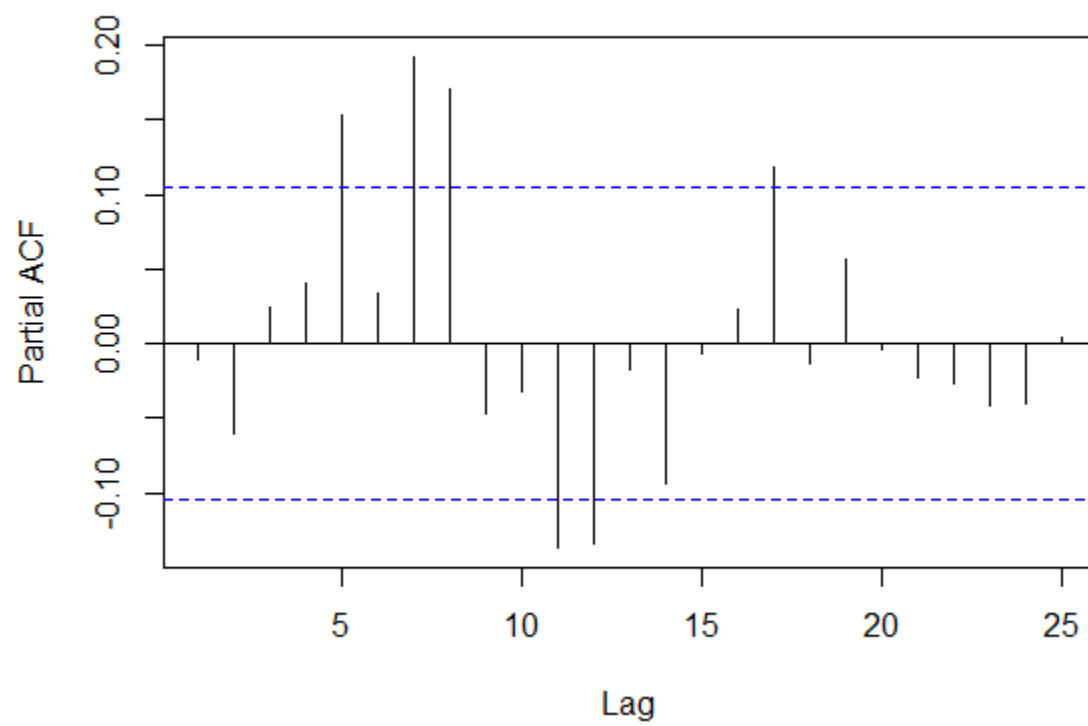


Figure 4.19: PACF of the Gamma Autoregressive Harmonic Model for Exchange rate

4.2.3 Model Comparison based on Model Validation and Forecast Evaluation

Metrics for Gamma Autoregressive Harmonic Model (Exchange Rate)

The derived Gamma Autoregressive Harmonic Model effectiveness is contrasted with the Autoregressive model and Harmonic model with regard to R^2 , $\bar{R}^2$ and forecast evaluation metrics. The R^2 and $\bar{R}^2$ values for the GARHM showed a better fit with good forecasting power (Table 4.3).

The developed GARHM, Autoregressive model and Harmonic model were compared based on their forecast evaluation metrics. Gamma Autoregressive Harmonic Model has lower values (Table 4.4). This makes the developed model forecast have better efficient and predictive power.

Table 4.3: Values of Coefficient and Adjusted Coefficient of Determination for Gamma Autoregressive Harmonic Model for Exchange rate

Model	R ²	Adjusted R ²
AR	0.9610	0.9509
Harmonic	0.0008671	-0.00779
GARHM	0.9813	0.9710

Source: Researcher's Computation, 2025

Table 4.4: Forecast Evaluation for Gamma Autoregressive Harmonic model for Exchange rate

Models	AIC	BIC	RMSE	MAE	MASE	MAPE	DW
AR	3586.6410	3598.2150	39.9794	12.9133	1.3315	0.05066	1.9888
Harmonic	4970.9360	4986.3680	290.3159	170.2260	19.9966	0.8433	0.01924
GARHM	3585.9030	3505.1930	39.7097	11.0847	1.2828	0.03613	2.01916

Source: Researcher's Computation, 2025

4.2.4 Forecast with Gamma Autoregressive Harmonic Model for Exchange Rate

The fitted Gamma Autoregressive Harmonic Model for the Nigerian exchange rate forecasted values for 30 months is given in Table 4.5 at a confidence of 95%. The forecasted values from March 2025 to August 2027 indicate a steady upward trend, with predictions ranging from 1503.475 in March 2025 to 1588.158 in August 2027. The 95% confidence interval suggests potential fluctuations, with lower bounds starting at 1425.197 and upper bounds reaching 2004.841. This highlights a consistent increase, reflecting the model's expectation of continued growth over time.

The fitted Gamma Autoregressive Harmonic Model for the Nigerian exchange rate from January 1996 to February 2025, which is a period of 29 years, is given in Figure 4.20. Nigeria's exchange rate will increase at a steady rate from March 2025 to August 2027, which indicates further depreciation of the Nigerian Naira over the forecasted period.

Table 4.5: Forecasted Values for Exchange Rate at 95% Confidence Interval

Months	Time Period	Forecasted Values	Lower CI (95%)	Upper CI (95%)
Mar 2025	351	1503.475	1425.197	1581.753
Apr 2025	352	1507.396	1396.804	1617.988
May 2025	353	1511.261	1375.948	1646.574
Jun 2025	354	1515.068	1358.977	1671.159
Jul 2025	355	1518.815	1344.473	1693.157
Aug 2025	356	1522.501	1331.708	1713.294
Sep 2025	357	1526.123	1320.247	1731.999
Oct 2025	358	1529.680	1309.807	1749.553
Nov 2025	359	1533.170	1300.190	1766.150
Dec 2025	360	1536.592	1291.252	1781.932
Jan 2026	361	1539.943	1282.882	1797.004
Feb 2026	362	1543.222	1274.996	1811.449
Mar 2026	363	1546.427	1267.524	1825.331
Apr 2026	364	1549.557	1260.411	1838.704
May 2026	365	1552.611	1253.610	1851.611
Jun 2026	366	1555.585	1247.082	1864.088
Jul 2026	367	1558.479	1240.794	1876.164
Aug 2026	368	1561.292	1234.718	1887.866
Sep 2026	369	1564.022	1228.828	1899.215

Months	Time Period	Forecasted Values	Lower CI (95%)	Upper CI (95%)
Oct 2026	370	1566.667	1223.103	1910.231
Nov 2026	371	1569.226	1217.522	1920.929
Dec 2026	372	1571.697	1212.070	1931.324
Jan 2027	373	1574.079	1206.730	1941.429
Feb 2027	374	1576.372	1201.489	1951.255
Mar 2027	375	1578.572	1196.333	1960.812
Apr 2027	376	1580.680	1191.252	1970.108
May 2027	377	1582.694	1186.235	1979.153
Jun 2027	378	1584.612	1181.272	1987.952
Jul 2027	379	1586.434	1176.355	1996.513
Aug 2027	380	1588.158	1171.475	2004.841

Source: Researcher's Computation, 2025

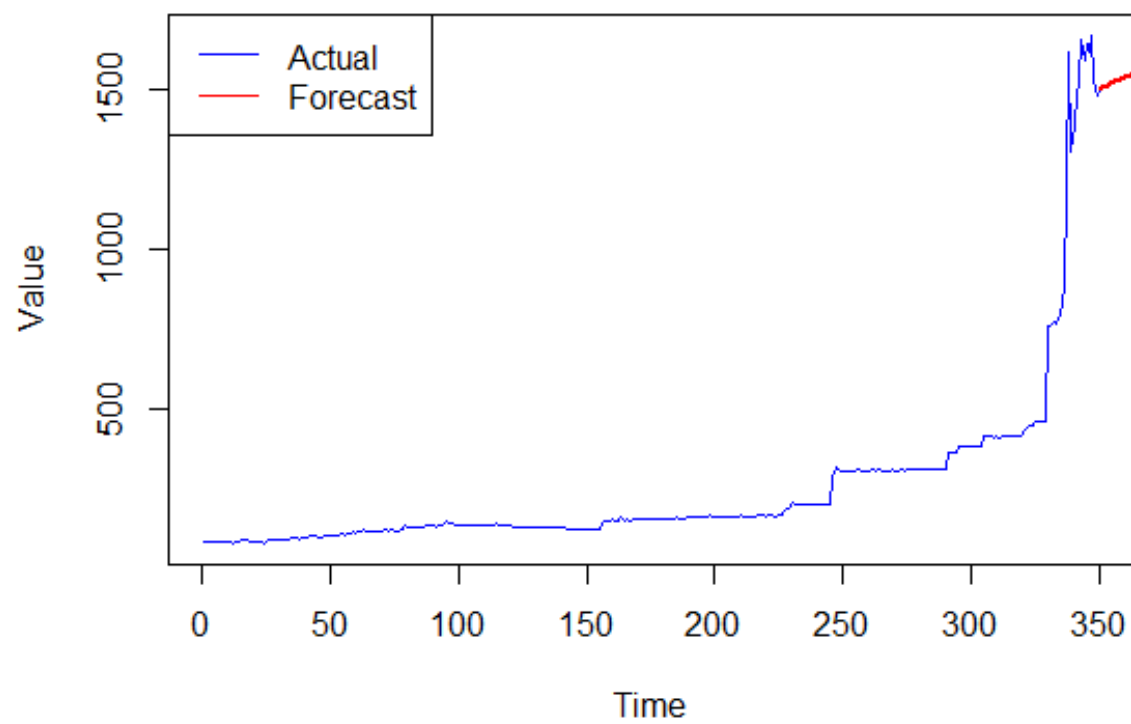


Figure 4.20: Forecast of Exchange Rate

4.2.5 Gamma Autoregressive Harmonic Model for CO₂ Emission

The analysis revealed a dominant period of approximately 11, suggesting a strong seasonal component within the time frame. This periodicity was subsequently incorporated into the hybrid GARHM in (4.2) to better capture both autoregressive dynamics and underlying oscillatory behavior. Table 4.6 indicates the order of the developed model. The lowest AIC and BIC values indicate that the optimal choice is that of order 3 across all the models.

The Gamma autoregressive harmonic model for estimated for CO₂ emission is given as;

$$\begin{aligned} y_t = & 26.99259651 + 0.22932417y_{t-1} - 0.13141193y_{t-2} + 0.21886704y_{t-3} \\ & - 0.06101750\sin 0.182\pi_1 t - 0.20387501\sin 0.182\pi_2 t \\ & - 0.15994324\sin 0.182\pi_3 t + 0.24103622\cos 0.182\pi_1 t \\ & - 0.08297314\cos 0.182\pi_2 t + 0.07511896\cos 0.182\pi_3 t \\ & + \varepsilon_t \end{aligned} \tag{4.2}$$

with $R^2 = 0.8992934$, *Adjusted* $R^2 = 0.8222824$ and Durbin Watson = 1.795396

Table 4.6: Order selection based on AIC and BIC values for CO₂ Emission

Models	AIC	BIC
AR (1)	651.7228	672.7292
AR (2)	650.7320	656.0079
AR (3)	650.4725	652.6425
Harmonic (1)	676.9560	682.6919
Harmonic (2)	679.8058	688.4097
Harmonic (3)	682.9147	694.3866
GARHM (1)	652.2897	659.4596
GARHM (2)	647.2171	658.6889
GARHM (3)	630.9437	651.0195

Source: Researcher's Computation, 2025

4.2.6 Gamma Autoregressive Harmonic Model Validation for CO₂ Emission

The R^2 in Table 4.7 revealed the variations in CO₂ emission as 89.9%. Additionally, the model has a high fit with an 82.2% power.

Figure 4.21 and the Jarque-Bera test (p-value = 0.00532) indicated that the residuals are not normally distributed. Figure 4.22 and Figure 4.23 showed that the residual does not have the usual structure and is significant. Considering Durbin-Watson statistics value, Figures 4.21, 4.22, and 4.23, the Gamma Autoregressive Harmonic Model is considered adequate for the CO₂ emission dataset.

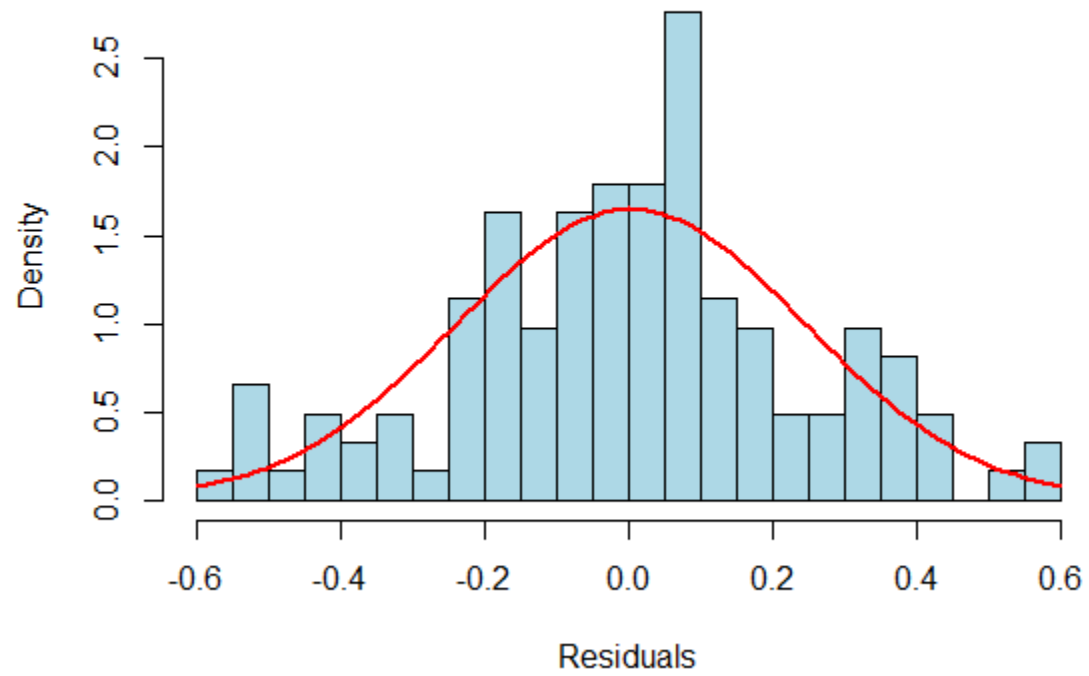


Figure 4.21: Histogram of Gamma Autoregressive Harmonic model for CO₂ Emission

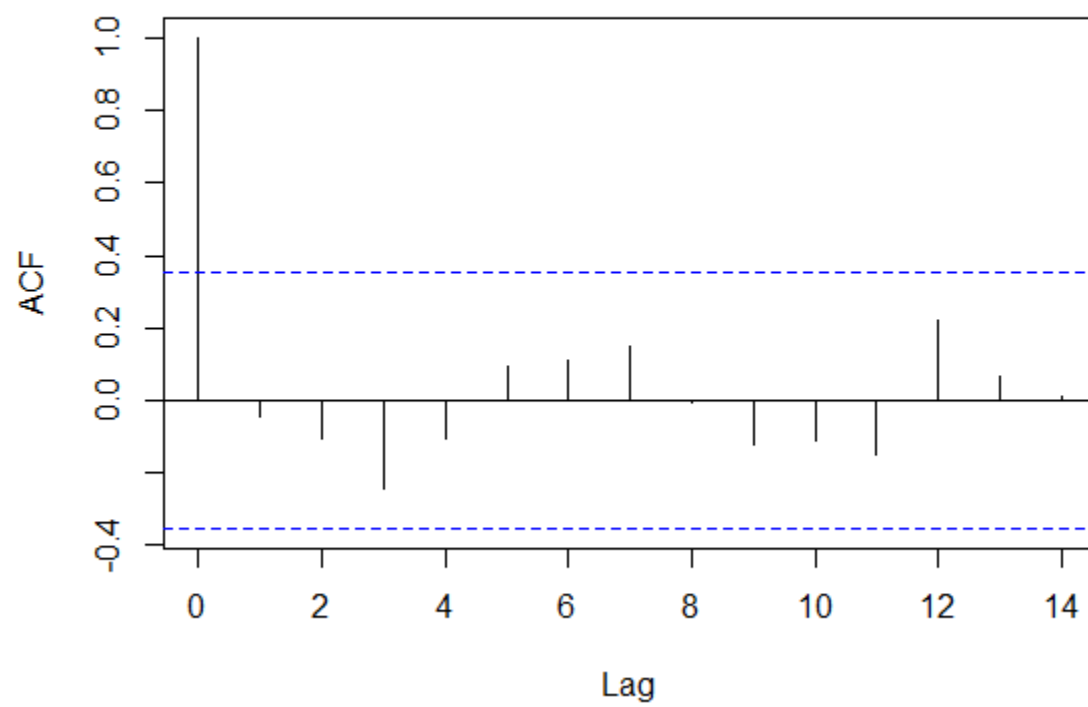


Figure 4.22: ACF of the Gamma Autoregressive Harmonic Model for CO₂ Emission

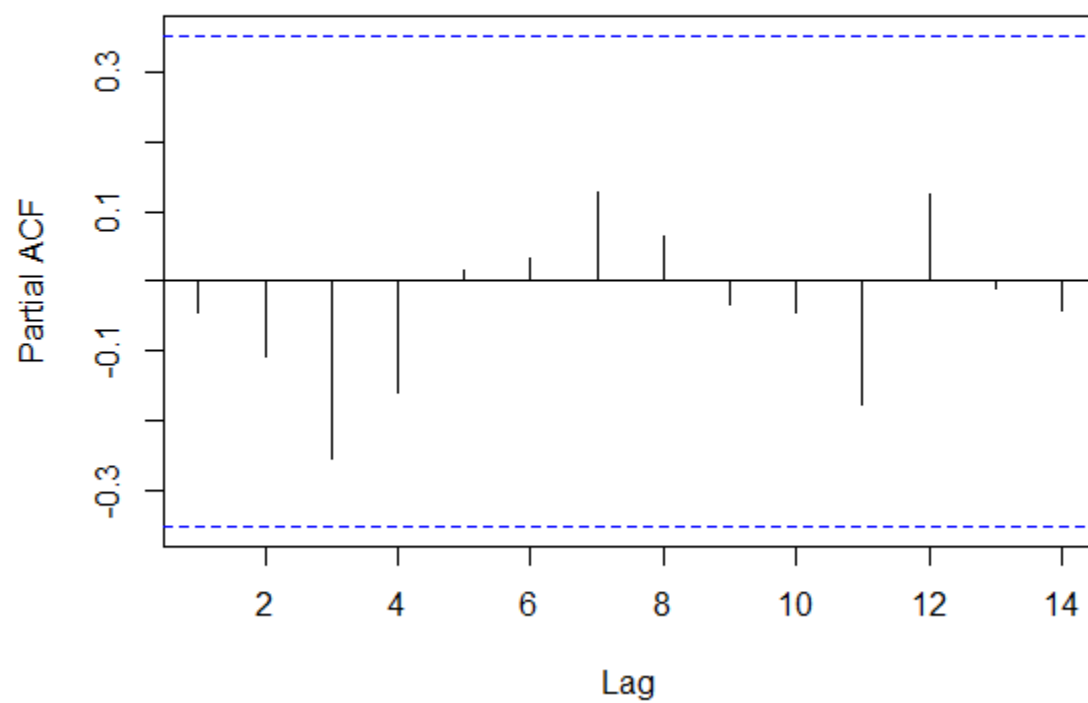


Figure 4.23: PACF of the Gamma Autoregressive Harmonic Model for CO₂ Emission

4.2.7 Model Comparison based on Model Validation and Forecast Evaluation

Metrics for Gamma Autoregressive Harmonic Model (CO₂ Emission)

The derived Gamma Autoregressive Harmonic Model effectiveness is contrasted with the Autoregressive model and Harmonic model. The R^2 and $\bar{R}^2$ values for the GARHM produced a better fit and forecasting ability (Table 4.7).

The developed GARHM, Autoregressive model and Harmonic model were compared based on their AIC, BIC, RMSE, MAE, MASE, and MAPE. Gamma Autoregressive Harmonic Model has better efficiency (Table 4.8).

Table 4.7: Values of Coefficient and Adjusted Coefficient of determination for Gamma Autoregressive Harmonic Model (CO₂)

Models	R ²	Adjusted R ²
AR	0.6291	0.5720
Harmonic	0.09075	-0.1859
GARHM	0.8992	0.8222

Source: Researcher's Computation, 2025

Table 4.8: Forecast Evaluation for Gamma Autoregressive Harmonic Model (CO₂)

Models	AIC	BIC	RMSE	MAE	MASE	MAPE	DW
AR	650.4725	657.6425	7254.2330	0.2001	0.9116	0.06722	2.0428
Harmonic	682.9147	694.3866	11358.8330	0.2976	1.3655	0.09978	0.3527
GARHM	630.9437	651.0195	3780.2600	0.1872	0.7670	0.03425	1.7953

Source: Researcher's Computation, 2025

4.2.8 Forecast with Gamma Autoregressive Harmonic Model for CO₂ emission

The fitted Gamma Autoregressive Harmonic (GARHM) model for CO₂ emissions in kilotons of forecasted values for 10 years is presented in Table 4.9 at a 95% confidence level. The forecasted values from 2021 to 2030 indicate a fluctuating trend, with predictions ranging from 109234.26 kilotons in 2021 to 95713.52 kilotons in 2030. The 95% confidence interval suggests possible variations, with lower bounds starting at 93479.52 kilotons and upper bounds reaching 122731.40 kilotons by the end of the forecast period.

The fitted Gamma Autoregressive Harmonic Model (GARHM) model for CO₂ emissions from 1990 to 2020, covering a period of 31 years (Figure 4.24). The out-of-sample forecast from 2021 to 2030 suggests a continued decrease in CO₂ emissions indicating a persistent downward trend over the forecast period.

Table 4.9: Forecasted Values for CO₂ Emission at 95% Confidence Interval

Years	Forecasted Values	Lower CI (95%)	Upper CI (95%)
2021	109234.26	93479.52	124989.00
2022	106897.89	86563.98	127231.80
2023	104878.78	81998.81	127758.80
2024	103107.96	78679.56	127536.40
2025	101534.18	76127.14	126941.20
2026	100120.91	74082.69	126159.10
2027	98843.62	72393.53	125293.70
2028	97687.40	70966.60	124408.20
2029	96644.72	69745.20	123544.20
2030	95713.52	68695.66	122731.40

Source: Researcher's Computation, 2025

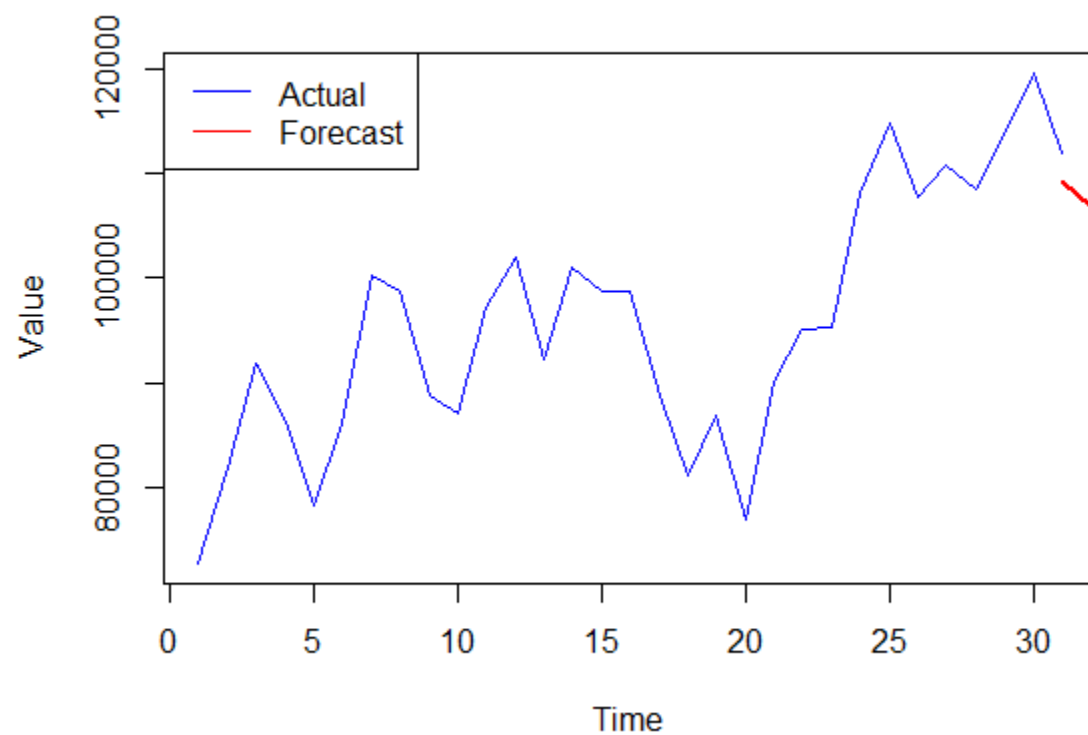


Figure 4.24: Forecast for CO₂ Emission

4.2.9 Gamma Autoregressive Harmonic Model for Temperature

The analysis revealed a dominant period of approximately 25, suggesting a strong seasonal component within this time frame. This periodicity was subsequently incorporated into the hybrid GARHM in (4.3) to better capture both autoregressive dynamics and underlying oscillatory behavior. Table 4.10 indicates the order of the developed model. The lowest AIC and BIC values indicate that the optimal choice is that of order 3 across all the models.

The Gamma Autoregressive Harmonic Model estimated for temperature is given as;

$$\begin{aligned} y_t = & 26.99259651 + 0.22932417y_{t-1} - 0.13141193y_{t-2} + 0.21886704y_{t-3} \\ & - 0.06101750\sin 0.08\pi_1 t - 0.20387501\sin 0.08\pi_2 t - 0.15994324\sin 0.08\pi_3 t \\ & + 0.24103622\cos 0.08\pi_1 t - 0.08297314\cos 0.08\pi_2 t \\ & - 0.07511896\cos 0.08\pi_3 t \\ & + \varepsilon_t \end{aligned} \tag{4.3}$$

with $R^2 = 0.79853$, $Adjusted R^2 = 0.78224$ and Durbin Watson = 1.944429

Table 4.10: Order selection based on AIC and BIC values for Temperature

Models	AIC	BIC
AR (1)	49.1792	57.6157
AR (2)	45.9650	57.2138
AR (3)	27.00422	41.0651
Harmonic (1)	103.9949	115.2437
Harmonic (2)	105.8868	122.7599
Harmonic (3)	108.3971	130.8946
GARHM (1)	44.3876	58.4485
GARHM (2)	37.3070	59.8044
GARHM (3)	21.2908	52.2248

Source: Researcher's Computation, 2025

4.2.10 Gamma Autoregressive Harmonic Model Validation for Temperature

The R^2 in Table 4.11 revealed that time explained the variations in temperature at 79.8%. Additionally, the model $\bar{R}^2$ value showed that the model has the goodness of fit of 78.2%.

Figure 4.25 and the Jarque-Bera test of 0.00522, indicated residuals non-normality. Figure 4.26 and Figure 4.27 indicated that the residual does not have the usual structure. Therefore, the Gamma Autoregressive Harmonic Model is considered adequate for the temperature dataset.

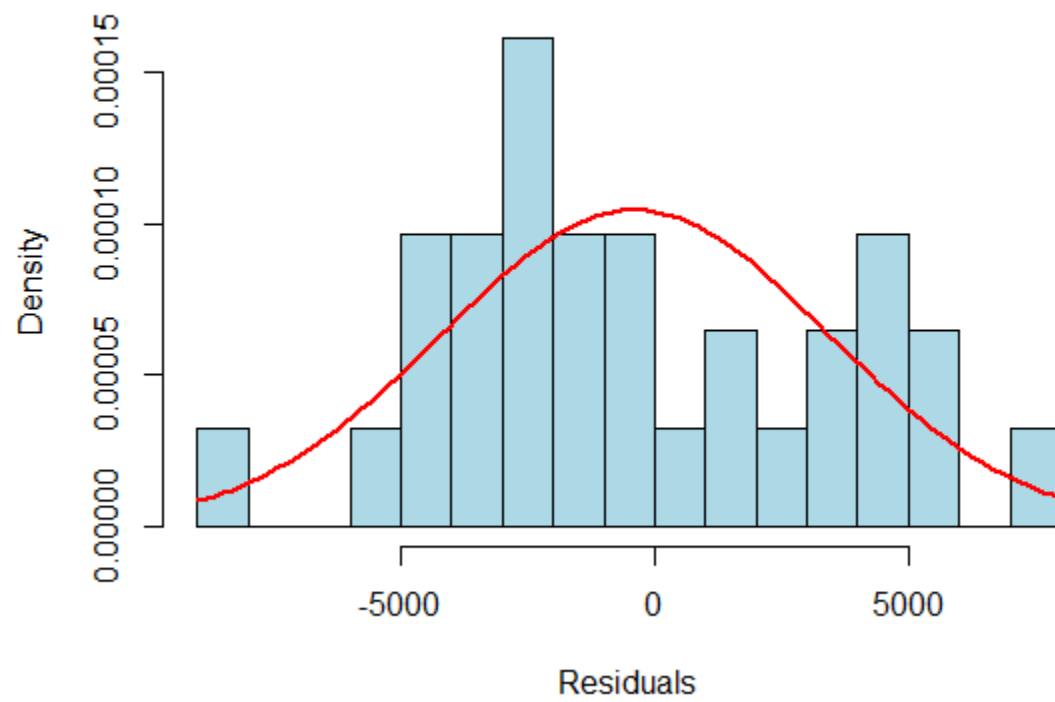


Figure 4.25: Histogram of Gamma Autoregressive Harmonic model for Temperature

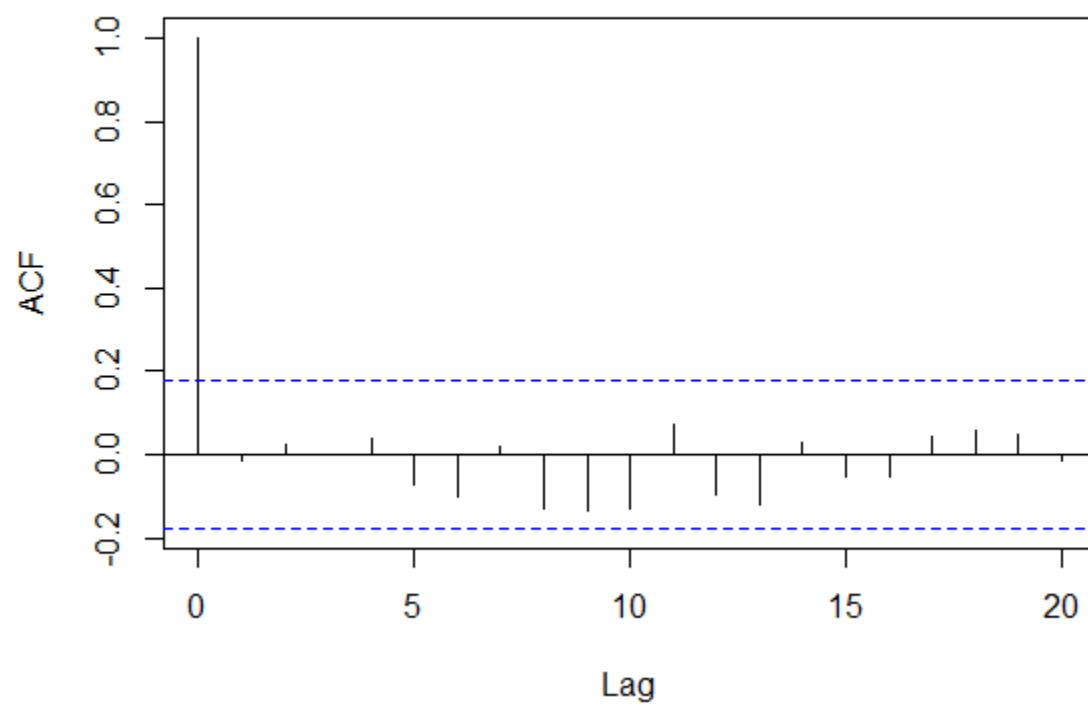


Figure 4.26: ACF of the Gaussian Autoregressive Harmonic Model for Temperature

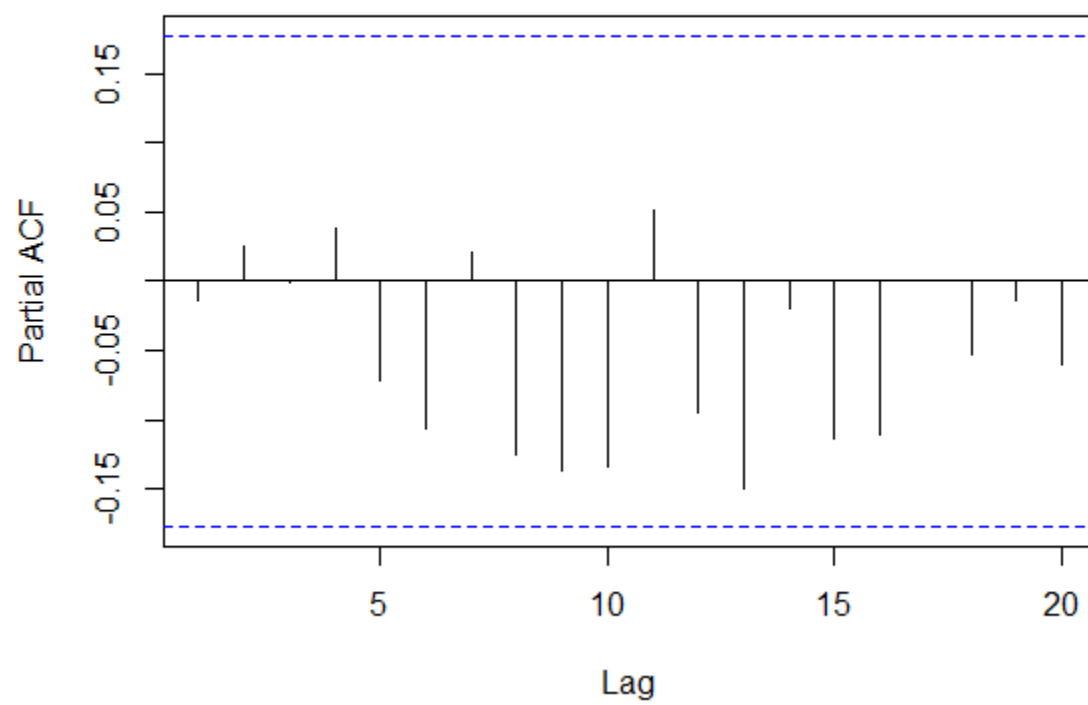


Figure 4.27: PACF of the Gaussian Autoregressive Harmonic Model for Temperature

4.2.11 Model Comparison based on Model Validation and Forecast Evaluation Metrics for Gamma Autoregressive Harmonic Model (Temperature)

The derived Gamma Autoregressive Harmonic Model effectiveness is contrasted with the Autoregressive model and Harmonic model. The R^2 value for the GARHM indicated better fit (Table 4.11) and the value of $\bar{R}^2$ signified that the GARHM performed better.

The developed GARHM, Autoregressive model and Harmonic model were compared. The forecast evaluation metrics of the Gamma Autoregressive Harmonic Model has lower values (Table 4.12). This makes the developed model have better forecast efficiency.

Table 4.11: Values of Coefficient and Adjusted Coefficient of determination for Gamma Autoregressive Harmonic Model (Temperature)

Models	R ²	Adjusted R ²
AR	0.6794	0.6012
Harmonic	0.08315	0.02734
GARHM	0.7985	0.7822

Source: Researcher's Computation, 2025

Table 4.12: Forecast Evaluation for Gamma Autoregressive Harmonic Model (Temperature)

Models	AIC	BIC	RMSE	MAE	MASE	MAPE	DW
AR	27.00422	53.0651	0.2580	0.2001	0.8197	0.007410	2.0770
Harmonic	108.3971	130.8946	0.3522	0.2898	1.1869	0.01072	0.8010
GARHM	21.2908	52.2248	0.2410	0.1872	0.7670	0.006940	1.9444

Source: Researcher's Computation, 2025

4.2.12 Forecast with Gamma Autoregressive Harmonic Model for Temperature

The fitted Gamma Autoregressive Harmonic Model (GARHM) for forecasting values from 2024 to 2033 is presented in Table 4.13 at a 95% confidence level. The forecasted values indicate a gradual decline, starting at 27.383 in 2024 and decreasing to 26.802 by 2033. The 95% confidence interval suggests minor fluctuations, with lower bounds ranging from 26.890 in 2024 to 26.285 in 2033, and upper bounds narrowing from 27.876 in 2024 to 27.319 in 2033. This trend highlights a steady downward movement, reflecting the model's expectation of a slight but consistent reduction over time.

The fitted Gamma Autoregressive Harmonic Model (GARHM) for global temperature from 1901 to 2023, covering a period of 123 years, is presented in Figure 4.28. The out-of-sample forecast from 2024 to 2033 indicates a steady decrease in temperature and also suggests continued global warming over the first few forecast periods or years.

Table 4.13: Forecasted Values for Temperature at 95% Confidence Interval

Years	Forecasted Values	Lower 95% CI	Upper 95% CI
2024	27.383	26.890	27.876
2025	27.074	26.569	27.580
2026	27.148	26.641	27.656
2027	27.097	26.583	27.611
2028	26.976	26.460	27.493
2029	26.943	26.426	27.459
2030	26.913	26.396	27.430
2031	26.861	26.344	27.378
2032	26.826	26.309	27.343
2033	26.802	26.285	27.319

Source: Researcher's Computation, 2025

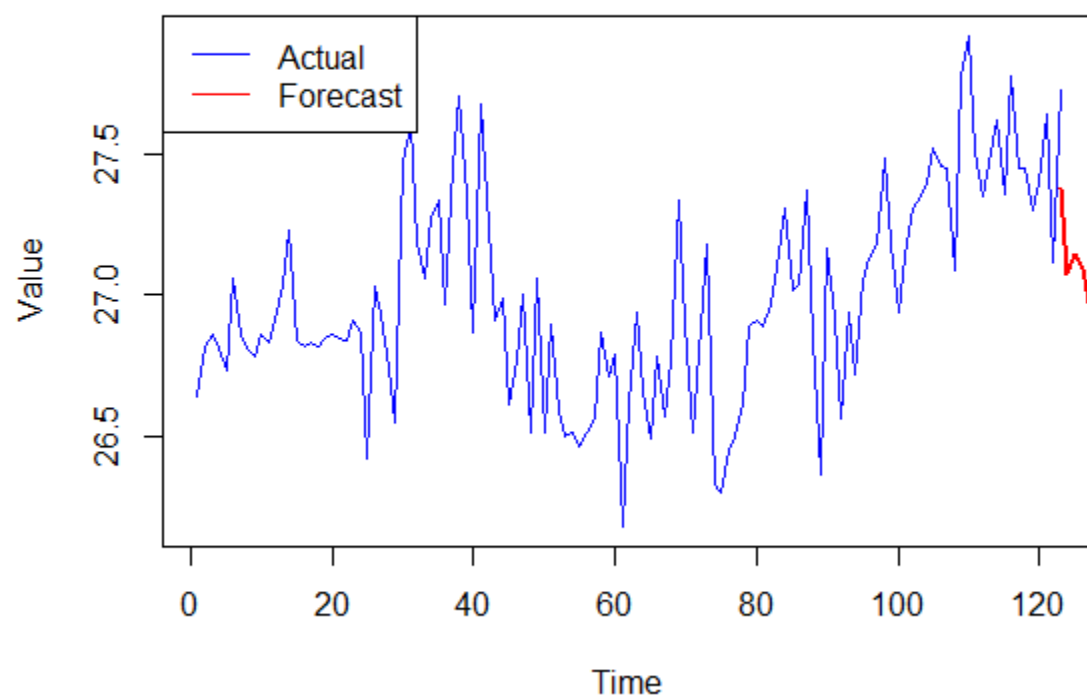


Figure 4.28: Forecast for Temperature

4.2.13 Gamma Autoregressive Harmonic Model for Inflation Rate

The analysis revealed a dominant period of approximately 4, suggesting a strong seasonal component within this time frame. This periodicity was subsequently incorporated into the hybrid GARHM in (4.4) to better capture both autoregressive dynamics and underlying oscillatory behavior. Table 4.14 indicates the order of the developed model. The lowest AIC and BIC values indicate that the optimal choice is that of order 1 across all the models.

The Gamma autoregressive harmonic model estimated for inflation rate is given as;

$$y_t = 17.7489297 - 0.0291613y_{t-1} + 5.3146135\sin 0.5\pi_1 t - 12.1443963\cos 0.5\pi_1 t + \varepsilon_t \quad (4.4)$$

with $R^2 = 0.72561$, *Adjusted* $R^2 = 0.70893$ and Durbin Watson = 2.003623

Table 4.14: Order selection based on AIC and BIC values for Inflation Rate

Models	AIC	BIC
AR (1)	609.4905	618.9199
AR (2)	613.4060	639.9785
AR (3)	617.3146	624.0303
Harmonic (1)	610.8136	619.3861
Harmonic (2)	614.5877	627.4465
Harmonic (3)	614.5877	627.4465
GARHM (1)	606.9292	617.6449
GARHM (2)	612.2156	629.3607
GARHM (3)	614.7685	630.3430

Source: Researcher's Computation, 2025

4.2.14 Gamma Autoregressive Harmonic Model Validation for Inflation Rate

The coefficient of determination (R^2) in equation (4.4) explained the variations in inflation rate as much as 72.5% with model's $\bar{R}^2$ value of 70.9%.

Figure 4.29 and the Jarque-Bera test (p-value of $2.2 \times e^{-16}$) indicated that the residuals non-normality. Figure 4.30 and Figure 4.31 showed that the residual does not have the usual structure. Therefore, the Gamma Autoregressive Harmonic Model is considered adequate for the inflation rate dataset.

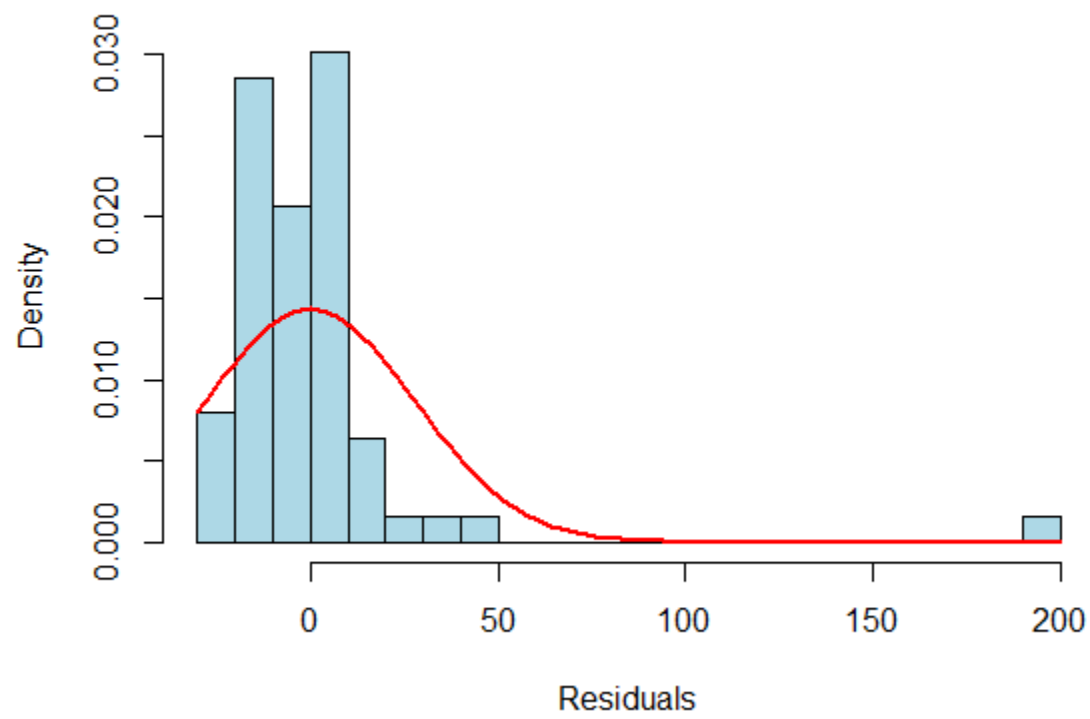


Figure 4.29: Histogram of the Gamma Autoregressive Harmonic Model for Inflation Rate

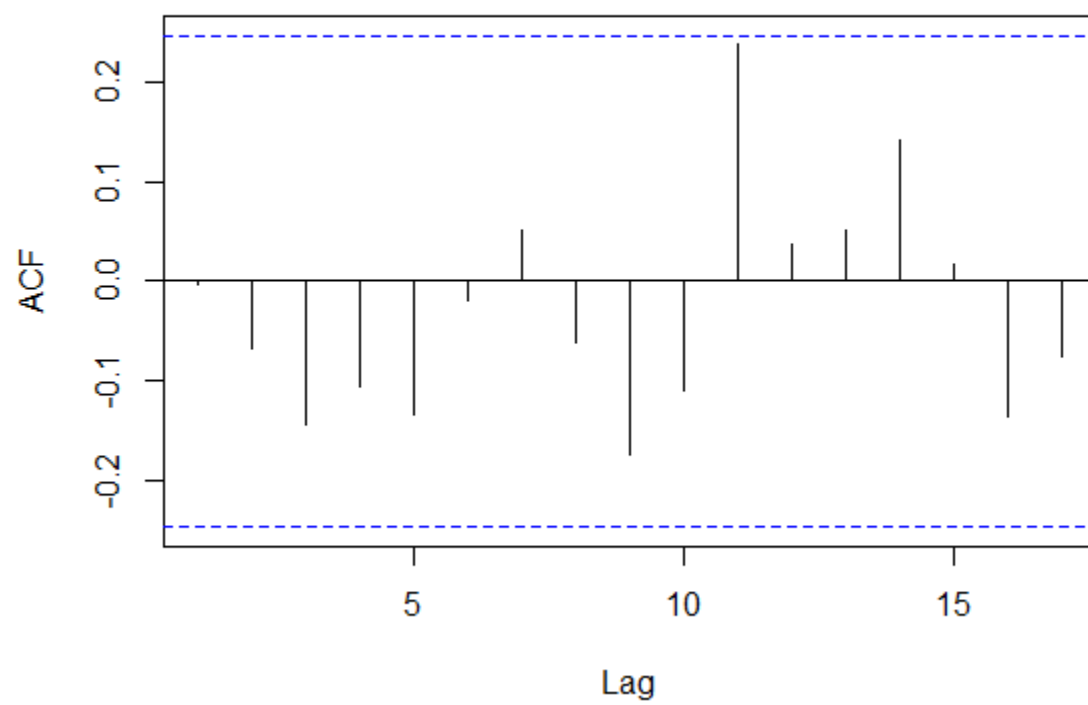


Figure 4.30: ACF of the Gamma Autoregressive Harmonic Model for Inflation Rate

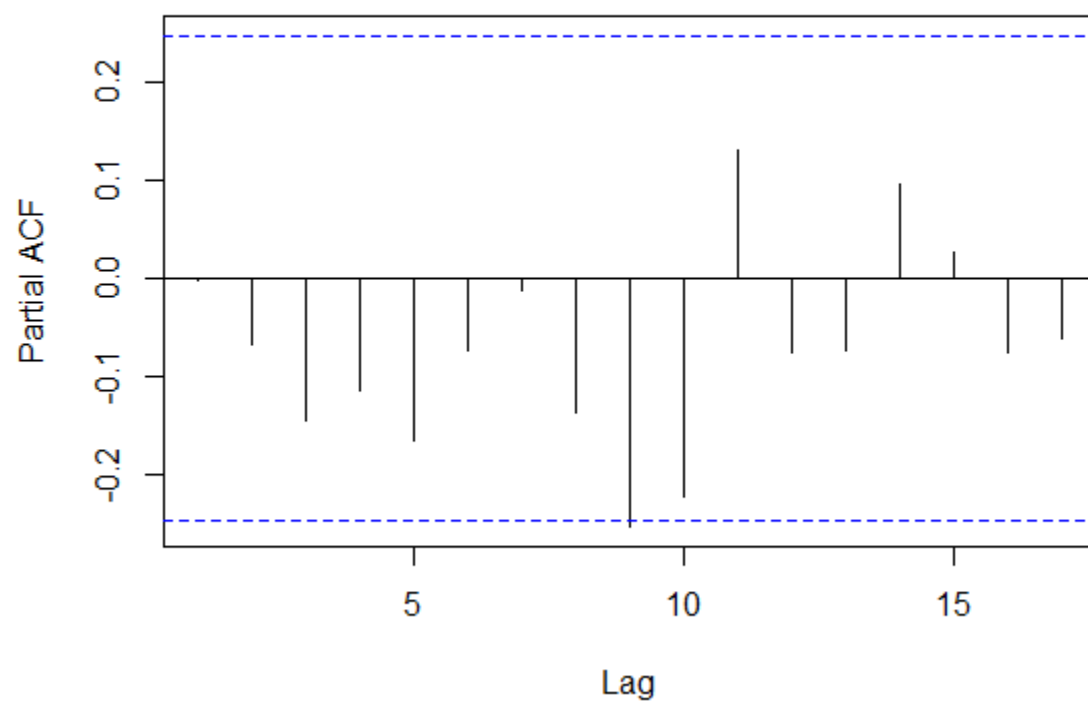


Figure 4.31: PACF of the Gamma Autoregressive Harmonic Model for Inflation Rate

4.2.15 Model Comparison based on Model Validation and Forecast Evaluation

Metrics for Gamma Autoregressive Harmonic model (Inflation Rate)

The derived Gamma Autoregressive Harmonic Model effectiveness is contrasted with the Autoregressive model and Harmonic model. GARHM showed a better fit (Table 4.15).

The developed GARHM, Autoregressive model and Harmonic model were compared based on their forecast evaluation metrics with Gamma Autoregressive Harmonic Model having lower values (Table 4.16). This makes the developed model forecast have better efficiency.

Table 4.15: Values of Coefficient and Adjusted Coefficient of determination for Gamma Autoregressive Harmonic Model (Inflation Rate)

Models	R ²	Adjusted R ²
AR	0.4775	-0.0275
Harmonic	0.005610	-0.3060
GARHM	0.7256	0.7089

Source: Researcher's Computation, 2025

Table 4.16: Forecast Evaluation for Gamma Autoregressive Harmonic Model (Inflation Rate)

Models	AIC	BIC	RMSE	MAE	MASE	MAPE	DW
AR	606.7600	618.9200	27.6210	14.1120	0.8726	2.3267	2.0006
Harmonic	609.4900	625.3300	28.0260	14.7290	0.9118	2.5103	1.9820
GARHM	604.9300	617.6400	14.5380	13.5410	0.8383	2.0369	2.0036

Source: Researcher's Computation, 2025

4.2.16 Forecast with Gamma Autoregressive Harmonic Model for Inflation Rate

The Gamma Autoregressive Harmonic Model forecasts inflation from 2024 to 2033 at a 95% confidence level in Table 4.17, suggesting an increasing trend, starting from 5.989 in 2024 to 15.624 in 2033. However, the wide confidence intervals (ranging from negative values to over 60). This reflects possible fluctuations and volatility in future inflation rates.

The fitted Gamma Autoregressive Harmonic Model (GARHM) for Nigeria's inflation rate from 1961 to 2023, covering a period of 63 years, is presented in Figure 4.32. The out-of-sample forecast from 2024 to 2033 indicates a steady increase in inflation, suggesting a continued increase in inflation over the forecast period.

Table 4.17: Forecasted Values for Inflation Rate at 95% Confidence Interval

Year	Forecasted Inflation	Lower 95% CI	Upper 95% CI
2024	5.989	-49.952	61.931
2025	6.904	-49.061	62.870
2026	7.710	-48.255	63.676
2027	8.623	-47.343	64.588
2028	9.625	-46.340	65.591
2029	10.709	-45.257	66.674
2030	11.862	-44.104	67.828
2031	13.074	-42.892	69.039
2032	14.332	-41.634	70.298
2033	15.624	-40.341	71.590

Source: Researcher's Computation, 2025

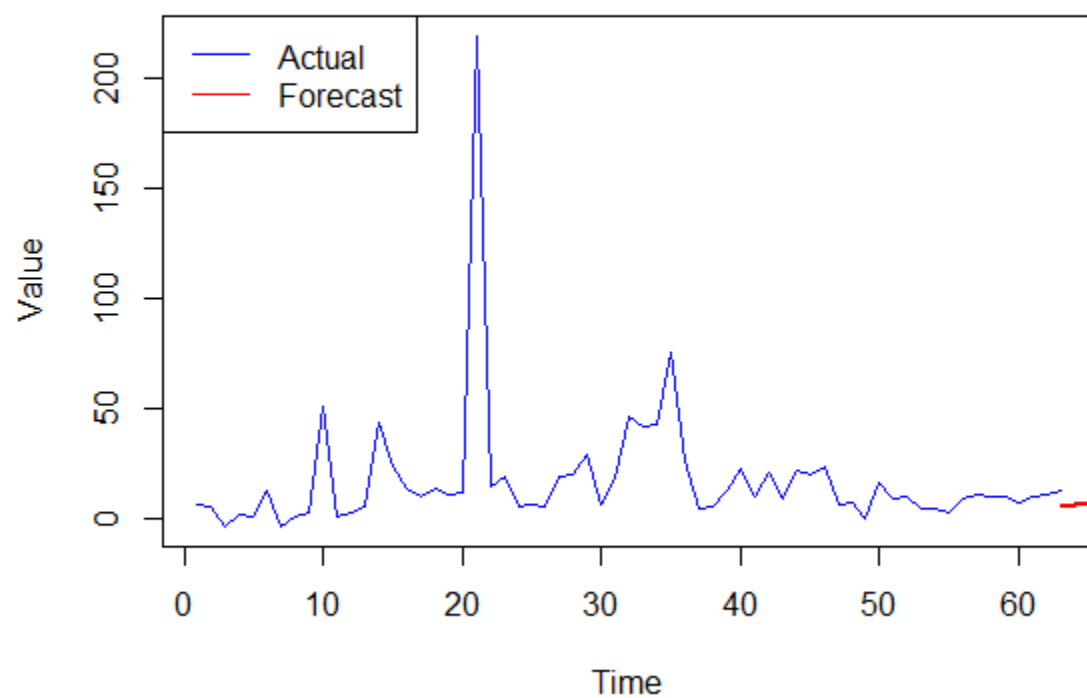


Figure 4.32: Forecast for Inflation Rate

4.3 Discussion

The variation exhibited by the Nigerian exchange rate was identified to exhibit skewness and seasonal variation (Olatayo and Ekerikevwe, 2022; Zamani *et al.*, 2021; Ahmed *et al.*, 2014). R^2 value demonstrated time accounted for variability in exchange rate in Nigeria to the tune of 98.1%. $\bar{R}^2$ value demonstrated the model fit is good with 97.1% level of predictive power. The price of the Nigeria exchange rate was predicted using the estimated Gamma Autoregressive Harmonic approach from March 2025 through August 2027, as the out-of-sample forecast indicated that Nigeria's exchange rate may increase over the period forecasted. The Gamma Autoregressive Harmonic Model (GARHM) was compared with the Autoregressive (AR) model and Harmonic model using R^2 , $\bar{R}^2$ and forecast evaluation metrics. The R^2 value showed that the developed GARHM model accounted for the variability in the Nigerian exchange rate better than the AR and Harmonic models. As well, $\bar{R}^2$ value showed the GARHM approach exhibited better goodness of fit in addition to higher forecasting ability than AR and Harmonic models.

Also, GARHM was utilized to model and forecast the yearly Nigeria temperature to demonstrate its effectiveness. From the analyzed results, the Durbin-Watson statistics showed no serial correlation. A Gamma Autoregressive Harmonic Model for the temperature series was fitted. According to the model, for every increment in time, there will be a unit decrease or increase in the temperature series on a yearly basis. The R^2 value was utilized to demonstrate that up to 79.8% of the variations in the temperature series in Nigeria could be explained by time. The model's goodness of fit was also demonstrated by the value with a 78.2% level of predictive ability. The developed (GARHM) approach was diagnosed to be suitable for forecasting Nigerian temperature series in terms of the histogram, ACF, and PACF of residuals. The Gamma Autoregressive

Harmonic Model (GARHM) was compared with the Autoregressive (AR) model and Harmonic model using R^2 , $\bar{R}^2$ and forecast evaluation metrics with GARHM showing better efficiency.

Furthermore, GARHM was utilized to model and forecast the yearly CO₂ emission in Nigeria in order to demonstrate its effectiveness. From the analyzed results, the Durbin-Watson statistics value indicated no serial correlation. A Gamma Autoregressive Harmonic Model equation for CO₂ emission series was fitted. According to the model, for every increment in time, there will be a unit decrease or increase in CO₂ emission on a yearly basis. The R^2 values were utilized to demonstrate that up to 89.9% of the variations in the CO₂ emission series in Nigeria could be explained by time. The model's goodness of fit was also demonstrated by the value $\bar{R}^2$ with an 82.2% level of predictive ability. The developed (GARHM) approach was diagnosed to be suitable for forecasting Nigerian CO₂ emission series in terms of histograms, ACF, and PACF of residuals.

Nigerian yearly macroeconomic variables (inflation rate) were analyzed as well using the developed Gamma Autoregressive Harmonic Model (GARHM). The time plot from 1960 to 2023 and the ADF test showed non-stationarity. The inflation rate series exhibited skewness and seasonal variations, and this was indicated by continuous rise and fall over the period under consideration (Olatayo and Ekerikevwe, 2022; Zamani *et al.*, 2021). The fitted Gamma Autoregressive Harmonic Model is considered adequate for modelling and forecasting Nigerian economic variables (Adedotun *et al.*, 2020; 2022). The derived Gamma Autoregressive Harmonic Model effectiveness was contrasted with the AR and Harmonic models with regard to R^2 , $\bar{R}^2$ and forecast evaluation metrics. The value of the coefficient of determination for the developed

Gamma Autoregressive Harmonic Model revealed that predictors explained the variations in the Nigerian inflation rate series better than all the other models fitted.

CHAPTER FIVE

SUMMARY, CONCLUSION AND RECOMMENDATIONS

5.1 Summary

The aspect is used to present and discuss the findings of the research for determining whether the aim and objectives of the research were fulfilled. This was carried out by analyzing some Nigerian time series datasets: the monthly exchange rate from January 1996 to February 2025, yearly climatic series (temperature) from 1901 to 2023, yearly time series data on CO₂ emission from 1990 to 2020, and yearly macroeconomic variables (inflation rate) from 1961 to 2023.

In this study, data exploration was carried out using a time plot and descriptive statistics; this revealed that the Nigerian exchange rate exhibited trend and seasonal variations simultaneously. The developed (GARHM) approach was diagnosed to be suitable for forecasting the Nigerian exchange rate in terms of the histogram, ACF, and PACF of residuals. There is no serial correlation, according to the Durbin-Watson statistics value for the estimated GARHM approach. The forecast evaluation metrics, that is, AIC, BIC, RMSE, MAE, MASE, and MAPE, indicated that the forecast evaluation of the Gamma Autoregressive Harmonic Model has lower values, and this made the GARHM model more efficient than all other models considered.

The R^2 value showed the developed GARHM model accounted for the variability in Nigeria's temperature better than the AR and Harmonic models. $\bar{R}^2$ value showed the GARHM approach exhibited better goodness of fit in addition to higher forecasting ability than AR and Harmonic models. As well, the forecast evaluation metrics, that is, AIC, BIC, RMSE, MAE, MASE, and MAPE, indicated that the forecast evaluation of the Gamma Autoregressive Harmonic Model was lower, and this made it better for forecasting Nigerian temperature than all other fitted models.

The Gamma Autoregressive Harmonic Model (GARHM) was compared with the Autoregressive (AR) model and Harmonic model using R^2 , $\bar{R}^2$ and forecast evaluation metrics. The R^2 value showed the developed GARHM model accounted for the variability in the Nigerian CO₂ emission rate better than AR and Harmonic models. As well, $\bar{R}^2$ value showed the GARHM approach exhibited better goodness of fit in addition to higher forecasting ability than AR and Harmonic models. As well, the forecast evaluation metrics, that is, AIC, BIC, RMSE, MAE, MASE, and MAPE, indicated that the forecast evaluation of the Gamma Autoregressive Harmonic Model was lower, and this made it better for forecasting the Nigerian CO₂ emission rate than all other fitted models.

A visual look at the series shows that there is a continuous rise and fall trend but with cycles that are completed over time. The estimated results show that for every increment in time, there will be a unit decrease or increase in the inflation rate on a yearly basis. The R^2 value signified that up to 72.5% of variability within the Nigerian inflation rate could be explained by time. Furthermore, a high goodness of fit for the model was signified by $\bar{R}^2$ value with a 70.9% level of predictive power. As well, $\bar{R}^2$ signified Gamma Autoregressive Harmonic Model has a better goodness of fit and predictive power ability than all other models fitted. The results of the forecast evaluation metrics (AIC, BIC, RMSE, MAE, MASE, and MAPE) further indicated that the developed Gamma Autoregressive Harmonic Model has a better forecast efficiency than all other fitted models since the Gamma Autoregressive Harmonic Model has lower values of forecast evaluation metrics.

5.2 Conclusion

The study developed the Gamma Autoregressive Harmonic Model, coupling the autoregressive model and harmonic models to handle skewed and seasonal series. The time plot is fundamentally used to establish the variation exhibited by the time series datasets.

The efficiency of the developed model was established by modelling and forecasting Nigerian exchange rate, CO₂ emissions, temperature, and inflation rate time series datasets. The Gamma Autoregressive Harmonic Model consistently outperformed traditional models (Autoregressive and Harmonic models) in terms of coefficient of determination, adjusted coefficient of determination, and forecast evaluation metrics. In essence, the developed Gamma Autoregressive Harmonic Model is suitable for modelling and forecasting time series datasets that exhibit skew and seasonal variations.

Conclusively, the following have been achieved to contribute to academic knowledge:

- (i) The developed Gamma Autoregressive Harmonic Model (GARHM) provides an improved alternative for analysing time series datasets with skew-seasonal variations.
- (ii) The GARHM demonstrates the ability to handle different time series components, including skew, seasonality, and irregularities.
- (iii) The integration of autoregressive and harmonic components enhances the capability of hybrid modelling in time series analysis.
- (iv) MLE and OLS were used for the parameters' estimation.
- (v) The GARHM was validated using residual diagnostics and forecast evaluation metrics, proving its superior predictive power over traditional models.

- (vi) The GARHM was successfully applied to forecast Nigeria's exchange rate, CO₂ emissions, temperature, and inflation rate, demonstrating its versatility in modelling various economic and environmental indicators.

5.3 Recommendations

The following recommendations were proposed:

- (i) Researchers and policymakers should adopt the Gamma Autoregressive Harmonic Model (GARHM) for analyzing time series data that exhibit skew and seasonal variations, as it has demonstrated superior performance compared to traditional models.
- (ii) Future studies should explore the application of GARHM to other time series datasets beyond exchange rate, temperature, CO₂ emissions, and inflation rate to further validate its effectiveness in diverse fields.
- (iii) Given that GARHM performed well in forecasting, government agencies and financial institutions should leverage its predictive capabilities for economic planning, environmental monitoring, and policy formulation.
- (iv) Since time series data characteristics may evolve over time, continuous monitoring and model re-evaluation are necessary to ensure the sustained accuracy and relevance of forecasts generated using GARHM.

A Gamma Autoregressive Harmonic Model (GARHM) was developed which addressed the problem encountered by the existing models in capturing Non-Gaussian, periodic, and seasonal data. The GARHM demonstrated the ability to handle different time series components, including skew, seasonality, and irregularities. Different estimation methods were adopted in the study. The developed model was tested with the data financial, economic, environmental and meteorological fields as this study showed that the developed model can be applied in any fields with the data that

possesses Non-Gaussian, periodic, and seasonal properties. Also, the study showed that developed model is capable of handling data of different sample sizes.

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